# Exponential growth of norms in semigroups of linear automorphisms and Hausdorff dimension of self-projective IFS.

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**Abstract**

Given a finitely generated semigroup $S$ of the (normed) set of linear maps of a vector space $V$ into itself, we find sufficient conditions for the exponential growth of the number $N(k)$ of elements of the semigroup contained in the sphere of radius $k$ as $k \to \infty$. We relate the growth rate $\lim_{k\to\infty} \log N(k)/\log k$ to the exponent of a zeta function naturally defined on the semigroup and, in case $S$ is a semigroup of volume-preserving automorpisms, to the Hausdorff and box dimensions of the limit set of the induced semigroup of automorphisms on the corresponding projective space.

## 1 Introduction

The asymptotic behaviour of the norms of products of some fixed finite set of square matrices has been extensively studied in the context of the theory of random matrices. In particular, in a celebrated paper [FK60], Furstenberg and Kesten proved that, given some finite number of square matrices $A_i$, under suitable conditions the norm of almost all products of $k$ of the $A_i$ grows as $\gamma^k$, where $\gamma$ is the Lyapunov exponent associated to the $A_i$. In this paper we address the subject from a different point of view, namely we consider *all* possible products of the $A_i$ and provide sufficient conditions for the existence and boundedness of $\lim_{k\to\infty} \log N(k)/\log k$, where $N(k)$ is the number of these products that lie inside the (closed) sphere of radius $k$. As

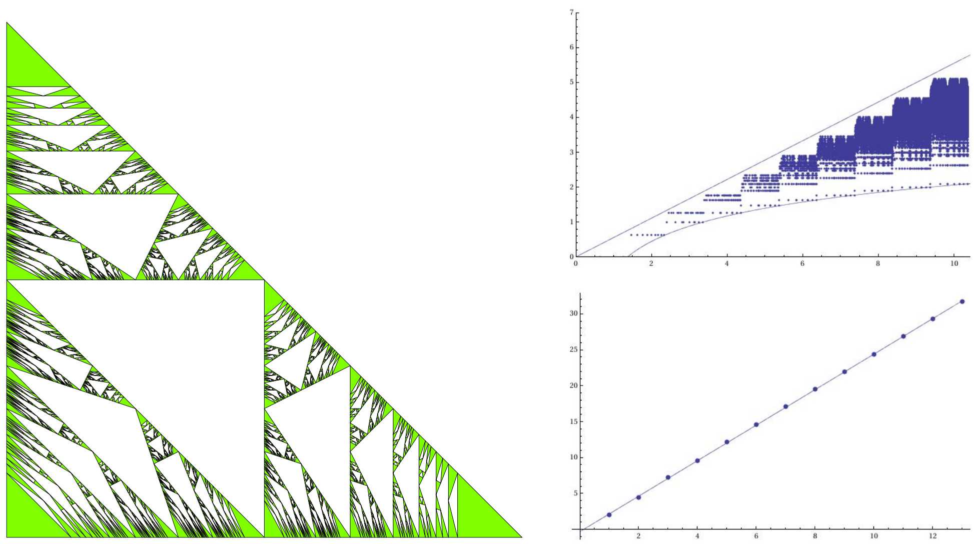

Figure 1: (left) The Cubic gasket $\boldsymbol{C}_3 \subset \mathbb{R}P^2$ in the triangle $T$ of vertices with homogeneous coordinates $[x : y : z] = [0 : 0 : 1]$, $[1 : 0 : 1]$, $[0 : 1 : 1]$. The picture shows (in green) the set $T_{7,\boldsymbol{C}_3}$ in the affine chart $z = 1$. (right, top) Log-log plot of the norms of matrices $C_I \in \boldsymbol{C}_3$, $|I| \leq 11$, ordered in lexicographic order. The fastest growing norms are $\|C_i \cdot C_{i+1} \cdots C_{i+k}\| \simeq \alpha_3^k$, where sums of indices are intended "modulo 3" and $\alpha_3 \simeq 1.84$ is the Tribonacci constant. The slowest growing ones are $\|C_i^k\| = k$. (right, bottom) Log-log plot of the function $N_{\boldsymbol{C}_3}(k)$ representing the number of matrices of $\boldsymbol{C}_3$ whose norm is not larger than $k$. Numerical data (the values of $N_{\boldsymbol{C}_3}(k)$ shown in the graphic are *exact*, see Table 3) indicate that $N_{\boldsymbol{C}_3}(k) \simeq Ak^s$ for $A \simeq 0.967$ and $s \simeq 2.444$. According to Conjecture 1, this entails that $\dim_B \boldsymbol{C}_3 \geq 1.63$.

a byproduct, we relate the rate of this growth to a zeta function naturally defined on semigroups of square matrices and, in particular cases, to the Hausdorff dimension of the limit set of the orbit of a point under the natural action induced by the semigroup on its corresponding projective space.

**Motivational Example 1: The *Cubic gasket* $\boldsymbol{C}_3$.** The real self-projective fractal $\boldsymbol{C}_3 \subset \mathbb{R}P^2$ (see Fig. 1) is the main reason for our interest in the subject of the present paper. It was first introduced, in the author's knowledge, by G. Levitt [Lev93] and independently rediscovered more recently by the author and I.A. Dynnikov in connection with the S.P. Novikov theory of plane sections of periodic surfaces [DD09]. We call it *cubic* because it is related to

the topology of plane sections of the cubic polyhedron $\{4,6|4\}$ (see [DD09] for details) and *gasket* because it has the same topology of the Sierpinski and Apollonian gaskets. Like the Sierpinski gasket, it can be thought as the set obtained by removing from the (projective) triangle $T(\mathcal{E})$ with vertices $[e_1]$, $[e_2]$, $[e_3]$, where $\mathcal{E}=\{e_1,e_2,e_3\}$ is any frame of $\mathbb{R}^3$, the (projective) triangle with vertices $[e_1+e_2]$, $[e_2+e_3]$, $[e_3+e_1]$ and repeating this procedure recursively on the three triangles left. We denote by $T_{k,\boldsymbol{C}_3}\subset T(\mathcal{E})$ the set obtained after repeating this procedure $k$ times. Clearly $\boldsymbol{C}_3=\cap_{k=1}^{\infty}T_{k,\boldsymbol{C}_3}$, i.e. we can get as close as we please to $\boldsymbol{C}_3$ by considering sets $T_{k,\boldsymbol{C}_3}$ with large values of $k$. $\boldsymbol{C}_3$ can also be characterized as the (unique) subset of the triangle with vertices $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$ which is invariant under the action of the (free) subsemigroup of $PSL_3(\mathbb{N})$ generated by the projective automorphsims $\psi_i$, $i=1,2,3$, induced by the following three $SL_3(\mathbb{N})$ matrices:

$$C_1=\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix},\ C_2=\begin{pmatrix}1&1&0\\0&1&0\\0&1&1\end{pmatrix},\ C_3=\begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix}.$$

By abuse of notation, we denote by $\boldsymbol{C}_3$ also the semigroup generated by the $C_i$. As a consequence of a conjecture of S.P. Novikov [Nov00], the set $\boldsymbol{C}_3$ is supposed to have Hausdorff dimension *strictly* between 1 and 2. What makes checking this conjecture non-trivial is that each $\psi_i$ has exactly one of the three vertices as fixed point and in that point it has Jacobian equal to $\mathbb{1}_3$, namely the iterated function system (IFS) $\{\psi_1,\psi_2,\psi_3\}$ is *parabolic* rather than *hyperbolic*[1].

No analytical bound for this fractal is known to date. In Section 4, based on numerical evidence and Theorems 4 and 5, valid for $2\times 2$ matrices, we conjecture that the box dimension of $\boldsymbol{C}_3$ is related by the growth rate of the norms of the elements of the semigroup generated by the $C_i$ (see Fig. 1), namely by $s=\lim_{k\to\infty}\log N(k)/\log k$, where $N(k)$ is the number of matrices of $\boldsymbol{C}_3$ inside the closed ball of $M_3(\mathbb{R})$ of radius $k$. According to this conjecture, $\dim_B \boldsymbol{C}_3\geq 2s/3\simeq 1.63$ (see Section 4.3).

**Motivational Example 2: The *Apollonian gasket.*** The complex self-projective fractal $\boldsymbol{A}_3\subset\mathbb{C}P^1$ is possibly the fractal with the oldest ancestry,

[1]Recall that a IFS $\{f_1,\dots,f_m\}$ on a metric space $(M,d)$ is said *hyperbolic* when all $f_i$ are contractions with respect to $d$ and *parabolic* when all $f_i$ are non-expanding maps.

since its construction relies on a celebrated result of the Hellenistic mathematician Apollonius of Perga (ca 262 BC – ca 190 BC), known in his times as *The Great Geometer*. Apollonius' result, contained in the now–lost book *Tangencies* but fortunately reported by Pappus of Alexandria in his *Collection* [Pap40] published about five centuries later, concerns the existence of circles tangent to a given triple of objects that can be any combination of points, straight lines and circles. In particular, given three circles which are mutually externally tangent to each other (sometimes called the *four coins problem* [Old96]), there exist exactly two new circles tangent to each of them, one externally and one internally (see Fig. 2). The three given circles plus any one of the new ones[2] form a *Descartes configuration*, since it was Descartes that stated the following remarkable relation between the curvatures $c_1, \dots, c_4$ of the four circles (see [Cox37] for details) memorialized three centuries later by the Chemistry Nobelist Frederick Soddy in his poem "The Kiss Precise" [Sod36] after rediscovering it independently: $2\sum_{i=1}^{4} c_i^2 = \left(\sum_{i=1}^{4} c_i\right)^2$.

Since Möbius transformations preserve circles and are transitive on triples of distinct points, they also act transitively on the set of all possible Descartes configurations; this fact suggests that their most natural environment is the Riemann sphere $\mathbb{C}P^1$ rather than the plane. Any Descartes configuration $D$ divides $\mathbb{C}P^1$ in 4 curvilinear triangles $T_i$ in such a way that every circle of $D$ is one of the two Soddy circles of the remaining three circles of $D$. By drawing the new Soddy circle of each of the 4 triples we are left with 4 new Descartes configurations. By repeating this process recursively we generate an infinite osculating circle packing of $\mathbb{C}P^1$ which, not surprisingly, is called *Apollonian packing*.

Here we rather focus our attention on any one of the curvilinear triangles $T$ and call *Apollonian gasket* $\boldsymbol{A}_3$ the set of points of $T$ left after removing from $T$ the interior of all Soddy circles inside it. Like in case of the cubic gasket, $\boldsymbol{A}_3$ can be characterized as the invariant set of a complex self-projective *parabolic* IFS. The fact that, thanks to the complex structure of $\mathbb{C}P^1$, $\boldsymbol{A}_3$ is *self-conformal* was exploited by Mauldin and Urbanski to prove some of its fundamental properties [MU98]. Unfortunately these techniques do not seem to extend to the previous (real) case, when the IFS maps are parabolic but not conformal.

In 1967 K.E. Hirst [Hir67] introduced the *Hirst semigroup* $\boldsymbol{H}$, namely the

[2] In Soddy's honor the two new circles are called *Soddy's circles.*

subsemigroup of $SL_4(\mathbb{N})$ generated by the matrices

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \; H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \; H_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$

as an effective tool to represent the radii of the Soddy circles in the gasket.

In a series of fundamental contributions to the study of the Hausdorff dimension of the gasket [Boy70, Boy71, Boy72, Boy73a, Boy73b, Boy82], D.W. Boyd ultimately characterized this dimension in terms of the Hirst semigroup by proving (implicitly, in terms of the circles' curvatures) that: 1) the number $N_{\boldsymbol{H}}(k)$ of the semigroup matrices with norm[3] not larger than $k$ is logarithmically asymptotic to $k^s$ for some $s > 0$, namely $\lim_{k\to\infty} \frac{\log N_{\boldsymbol{H}}(k)}{\log k} = s$; 2) $\dim_H \boldsymbol{A}_3 = s$.

Later in this paper we show that $\boldsymbol{A}_3$ can be seen as the invariant set of the *parabolic* Kleinian IFS corresponding to the subsemigroup of $SL_2(\mathbb{C})$ generated by the matrices

$$A_1 = \begin{pmatrix} 0 & i \\ i & 2 \end{pmatrix}, \; A_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}, \; A_3 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix},$$

which by abuse of notation we will denote too by $\boldsymbol{A}_3$, and conjecture that $\lim_{k\to\infty} \log N_{\boldsymbol{A}_3}(k)/\log k = 2\dim_H \boldsymbol{A}_3$ based on the fact that this relation holds for similar semigroups that induce *hyperbolic* IFSs and on numerical evidence.

**Motivational Example 3: The *Sierpinski gasket.*** The self-affine fractal set $\boldsymbol{S}_3 \subset \mathbb{R}^2$ is less ancient than the Apollonian one, having been introduced in the Mathematics literature by W. Sierpinski only in 1915 [Sie15], but it does have a long history too since its pattern has been known and used in art for about a millennium [PA02] (see Fig. 3). Its dimension is easily calculated: $\dim_H \boldsymbol{S}_3 = \log_2 3$ (e.g. see [Fal90]). Since both $PSL_3(\mathbb{R})$ and $PSL_2(\mathbb{C})$ contain a subgroup isomorphic to the group of affine transformations of the plane, the Sierpinski gasket can also be seen as a real (respectively complex) self-projective fractal of $\mathbb{R}P^2$ (respectively $\mathbb{C}P^1$).

[3]Since all norms are equivalent in finite dimension, this is true for any norm.

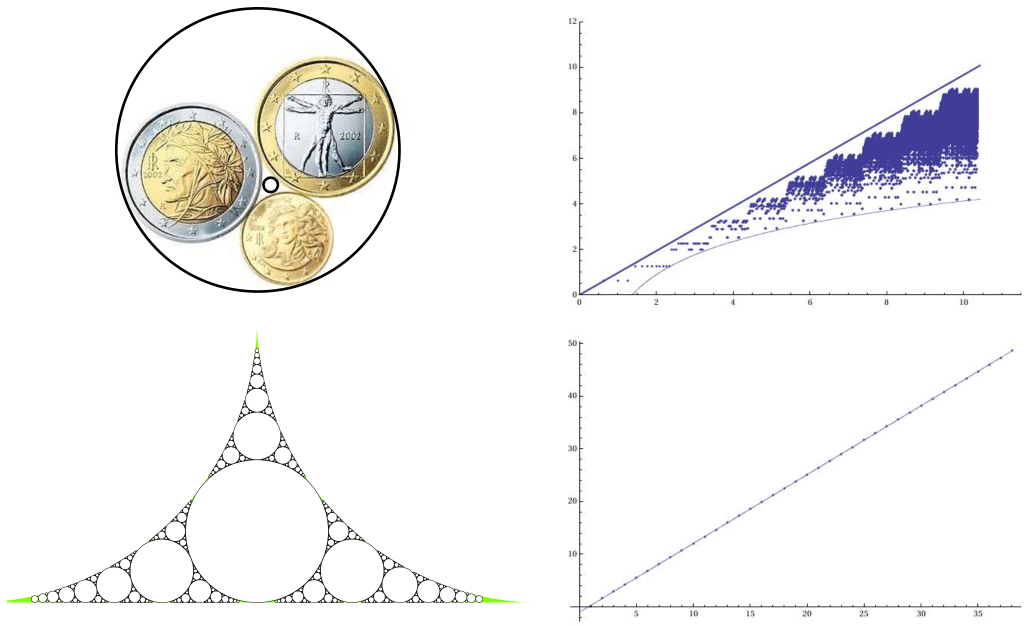


Figure 2: (top, left) Inscribed and circumscribed circles in the *four coins problem*. (bottom, left) Apollonian gasket $\boldsymbol{A}_3$ in the curvilinear triangle with mutually tangent arcs as sides and the points of homogeneous coordinates $[z : w]$ equal to $[1 : 1]$, $[-1 : 1]$, $[i, 1]$ as vertices, represented in the affine chart $w = 1$. In the image it is shown (in green) the set $T_{7,\boldsymbol{A}_3}$. (right) Log-log plots of the norms of the matrices of the semigroup $\boldsymbol{H}$ in lexicographic order (top) and of the relative function $N_{\boldsymbol{H}}(k)$ (bottom) counting the number of matrices of $\boldsymbol{H}$ whose norm is not larger than $k$ (see Table 3 for the values of $N_{\boldsymbol{H}}(k)$ shown in the graph).

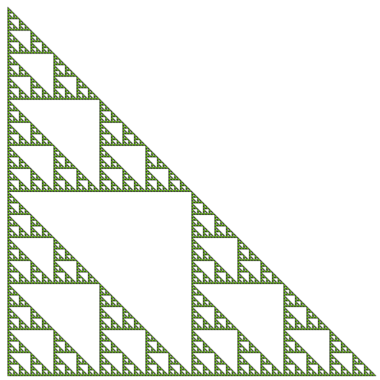


Figure 3: (left) Image of the Sierpinski gasket in the triangle $T$ with vertices $(0,0)$, $(1,0)$, $(0,1)$. In the picture it is shown, in green, the set $T_{7,\boldsymbol{S}_3}$. (right) Detail of a cosmatesque [PA02] mosaic dated about 11th–12th century (photo taken by the author at the Phillips Museum in Washington DC).

A semigroup generating the Sierpinski fractal in $\mathbb{R}P^2$ is, for example, the one generated by the matrices

$$S_1^{\mathbb{R}} = \frac{1}{\sqrt[3]{2}}\begin{pmatrix}2&0&0\\1&1&0\\1&0&1\end{pmatrix},\; S_2^{\mathbb{R}} = \frac{1}{\sqrt[3]{2}}\begin{pmatrix}1&1&0\\0&2&0\\0&1&1\end{pmatrix},\; S_3^{\mathbb{R}} = \frac{1}{\sqrt[3]{2}}\begin{pmatrix}1&0&1\\0&1&1\\0&0&2\end{pmatrix}.$$

One generating it in $\mathbb{C}\mathrm{P}^1$ is, for example, the one induced by the matrices

$$S_1^{\mathbb{C}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&i\\0&2\end{pmatrix},\; S_2^{\mathbb{C}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\0&2\end{pmatrix},\; S_3^{\mathbb{C}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\0&2\end{pmatrix}.$$

The regularity of the matrices $S_i^{\mathbb{R}}$ and $S_i^{\mathbb{C}}$ makes possible to perform simple direct calculations that illustrate the main points of this paper.

Consider first the real version. Let $\|S\|_\infty$ be the norm given by the maximum absolute row sum of $S$. Since all lines of the $S_i^{\mathbb{R}}$ sum to 2 then $\|S_{i_1}^{\mathbb{R}}\cdots S_{i_p}^{\mathbb{R}}\| = (2^{-1/3})^p \cdot 2^p = 2^{2p/3}$ for every $p \geq 1$. Hence in the sphere of radius $k$ lie

$$N(k) = \sum_{p=0}^{\lfloor \frac{3}{2}\log_2 k\rfloor} 3^p = \frac{3^{\lfloor \frac{3}{2}\log_2 k+1\rfloor}-1}{2}$$

products of the $S_i^{\mathbb{R}}$, where $\lfloor 3\log_2 k/2 \rfloor$ is the integer part of $3\log_2 k/2$, and therefore

$$s_{\mathbb{R}} = \lim_{k\to\infty} \frac{\log_2 N(k)}{\log_2 k} = \frac{3}{2}\log_2 3.$$

Note that $s_{\mathbb{R}}$ is also the exponent that separates the values for which the series $\sum_{S\in\langle S_i^{\mathbb{R}}\rangle} \|S\|^{-s}$ diverges from those for which it diverges, where the sum is extended to all elements of the semigroup freely generated by the $S_i^{\mathbb{R}}$. Finally, note that the following relation holds between the Hausdorff dimension of the Sierpinski gasket and the rate growth: $3\dim_H \boldsymbol{S}_3 = 2s_{\mathbb{R}}$.

Consider now the complex version. Endow $M_2(\mathbb{C})$ with the norm $\|S\|$ given by the largest modulus of the entries of $S$. Since the last row of each $S_i^{\mathbb{C}}$ is $(0,2)$, then $\|S_{i_1}^{\mathbb{C}} \cdots S_{i_p}^{\mathbb{C}}\| = (2^{-1/2})^p \cdot 2^p = 2^{p/2}$ for every $p \geq 1$. Hence in this case

$$N(k) = \sum_{p=0}^{\lfloor 2\log_2 k \rfloor} 3^p = \frac{3^{\lfloor 2\log_2 k+1 \rfloor} - 1}{2}$$

and therefore

$$s_{\mathbb{C}} = \lim_{k\to\infty} \frac{\log_2 N(k)}{\log_2 k} = 2\log_2 3.$$

Similarly to what happens in the real case, $s_{\mathbb{C}}$ is also the exponent that separates the values for which the series $\sum_{S\in\langle S_i^{\mathbb{C}}\rangle} \|S\|^{-s}$ diverges from those for which it diverges, where the sum is extended to all elements of the semigroup freely generated by the $S_i^{\mathbb{C}}$. Note that in this case the relation between the Hausdorff dimension of the Sierpinski gasket and the norms' growth rate is the following: $2\dim_H \boldsymbol{S}_3 = s_{\mathbb{C}}$.

For thorough surveys on the Sierpinski gasket and especially on the more challenging Apollonian gasket we refer the reader to the book by A.A. Kirillov [Kir07], the series of papers by Lagarias, Mallows, Wilks and Yan [GLM+03, GLM+05, GLM+06] and the recent article by Sarnak [Sar11].

The present paper is structured in the following way.

In Section 3 we generalize Boyd's arguments on the asymptotics of the sequence of radii of Soddy's circles in a Apollonian gasket and use them to obtain similar results on the asymptotics of norms of matrices in subsemigroups $S \subset M_n(K)$, $K = \mathbb{R}, \mathbb{C}$, by introducing a sufficient condition for the existence of $\lim_{k\to\infty} \log N_S(k)/\log k$, where $N_S(k)$ is the number of matrices in $S$ whose norm is not larger than $k$, and relating this limit to the critical exponent of a natural zeta-function defined on $S$.

In Section 4 we consider the action induced by these semigroups on the corresponding real or complex projective spaces and study, in particular but significant cases, the relation, observed above in case of the Sierpinski gasket, between the critical exponent of the semigroup and the Hausdorff (for $n = 2$) or box dimension (for $n \geq 3$) of the limit set of a point under its action.

We use Section 2 below to introduce the main concepts, notations and definition used throughout the paper and to state the main results of the paper.

# 2 Notations, definitions and main results.

**Matrices and Norms.** We endow the vector space $M_n(K)$ of all $n \times n$ matrices with coefficients in $K$ with the max norm, namely, given a matrix $M = (M^i_j)$,

$$\|M\| = \max_{i,j=1,\dots,n} \left\{ \left| M^i_j \right| \right\} .$$

We denote by $B^n_r \subset M_n(K)$ the closed ball of radius $r > 0$ in this norm. Note that this norm is not sub-multiplicative but rather

$$\sup_{P,Q \in M_n(K)} \frac{\|PQ\|}{\|P\| \|Q\|} = n \,. \tag{1}$$

Since in finite dimension all norms are equivalent, the main results of the paper will not depend on this particular choice.

**The multi-indices semigroup.** We denote by $\mathcal{I}^m$ the infinite $m$-ary tree of multi-indices of integers ranging from 1 to $m$ defined as follow. The root of the tree is the number 0. The $m$ children (*1-indices*) of 0 are the integers from 1 to $m$. Their children (*2-indices*) are the ordered pairs $1i, \dots, mi$ and so on recursively for the *k-indices*, $k > 2$. We denote by $\mathcal{I}^m_k$ the set of all $k$-indices of $\mathcal{I}^m$. Since we will use them often, we denote by $\mathcal{D}^m_\ell$, $\ell \geq 0$, the set of all *diagonal* multi-indices $I = i_1 \dots i_k \in \mathcal{I}^m$, $k \leq \ell$, i.e. such that $i_1 = \dots = i_k$, and set $\mathcal{D}^m = \cup_{\ell \geq 0} \mathcal{D}^m_\ell$. Similarly, we denote by $\mathcal{J}^m_\ell$, $\ell \geq 2$, the set of all *next-to-diagonal* multi-indices $I = i_1 i_2 \dots i_k \in \mathcal{I}^m$, $k \leq \ell$, i.e. those such that $i_1 \neq i_2 = \dots = i_k$, and set $\mathcal{J}^m = \cup_{\ell \geq 2} \mathcal{J}^m_\ell$.

We endow $\mathcal{I}^m$ with the canonical structure of semigroup given by $i_1 \dots i_k \cdot i'_1 \dots i'_{k'} = i_1 \dots i_k i'_1 \dots i'_{k'}$ with 0 as identity element. We also endow $\mathcal{I}^m$ with a partial order by saying that $I \geq J$ if $I$ can be factorized as $I = LJ$ for some

multi-index $L \neq 0$. Finally, we denote by $I' = i_1 \dots i_k$ the $k$-index obtained from the $(k+1)$-index $I = i_0 i_1 \dots i_k$ by dropping the first index on the left.

**Gaskets of matrices.** Given $m$ matrices $A_1, \dots, A_m \in M_n(K)$, $K = \mathbb{R}$ or $\mathbb{C}$, we denote by $\boldsymbol{A} = \langle A_1, \dots, A_m \rangle$ the semigroup they generate, given by the intersection of all subsemigroups of $M_n(K)$ containing all the $A_i$ and the unit matrix.

In this paper we are mainly interested in the asymptotic growth of norms of matrices in free semigroups but, since our results hold for the more general case when there are relations between the generators, we often formulate theorems using the more general concept of semigroup homomorphisms $\mathcal{I}^m \to M_n(K)$. We often denote such objects with the letter $\mathcal{A}$ and use the notation

$$A_I \stackrel{\text{def}}{=} \mathcal{A}(i_1 \dots i_k) = A_{i_1} \cdot \dots \cdot A_{i_k},$$

where $A_i \stackrel{\text{def}}{=} \mathcal{A}(i)$. We say that the matrices $A_1, \dots, A_m$ *generate* $\mathcal{A}$.

Notice that, when there is no relation between the $A_i$, then there is a bijection between $\mathcal{A}(\mathcal{I}^m)$ and $\{\mathbb{1}_n\} \cup \boldsymbol{A}$, so when the $A_i$ are free generators it is essentially equivalent referring to either the semigroup homomorphism $\mathcal{A}$ or the semigroup $\boldsymbol{A}$.

Given any $M \in GL_n(K)$ we denote by $\mathcal{A}_M$ the "right coset" map defined by $\mathcal{A}_M(I) \stackrel{\text{def}}{=} A_I M$. If $\boldsymbol{A}$ is free then there is a bijection between $\mathcal{A}_M(\mathcal{I}^m)$ and $\{M\} \cup \boldsymbol{A}M$, where $\boldsymbol{A}M$ is a right coset of $\boldsymbol{A}$. Clearly $\mathcal{A}_{\mathbb{1}_n} = \mathcal{A}$.

**Definition 1.** *We denote by $N_{\mathcal{A}_M}(r)$ the cardinality of the set $B^n_r \cap \mathcal{A}_M(\mathcal{I}^m)$ and say that $\mathcal{A}_M$ is a $m$-*gasket *(or simply a* gasket*) if $N_{\mathcal{A}_M}(r) < \infty$ for every $r > 0$. We say that the gasket $\mathcal{A}_M$ is* hyperbolic *if the sequence $a_k = \min_{I \in \mathcal{I}^m_k} \|A_I M\|$ diverges exponentially, namely if there exists $\alpha > 1$ such that $a_k \asymp \alpha^k$, where $\asymp$ means that the ratio of the terms on either side is bounded away from 0 and $\infty$ for all $k$. When $a_k$ is slower than exponential we say that $\mathcal{A}_M$ is* parabolic[4].

**Example 1.** *Every semigroup $\mathcal{A} : \mathcal{I}^m \to GL_n(\mathbb{C})$ whose generators have all their spectrum outside the unit circle is a hyperbolic gasket. Consider for example the simple case of $\mathcal{A} : \mathcal{I}^2 \to GL_2(\mathbb{C})$ with*

$$A_1 = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix},$$

[4] Note that $a_k$ cannot be faster than exponential so this covers all possible cases.

*where* $|\lambda| > 1$. *Then* $\min_{|I|=2k} \|A_I\| = \|A_1^k A_2^k\| = \|\lambda\|^k$.

**Example 2.** *Every semigroup* $\mathcal{A} : \mathcal{I}^m \to GL_n(\mathbb{C})$ *whose generators have all non-zero coefficients strictly larger than 1 is a hyperbolic gasket, since the norm of* $k$ *of such matrices will be not smaller than the* $k$*-th power of their smallest non-zero entry.*

**Example 3.** *The (free) semigroup* $\boldsymbol{C}_2 \subset SL_2(\mathbb{N})$ *generated by the two* parabolic *(i.e. with trace equal to* $\pm 2$*) matrices*

$$C_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

*is a parabolic gasket. It is a gasket because, if* $M \in SL_2(\mathbb{N})$ *is distinct from the identity, then* $\|C_i M\| \geq \|M\| + 1$ *since* $M$ *has at least a column with two entries different from 0. It is parabolic because*

$$\min_{I \in \mathcal{I}_k^m} \{\|C_I\|\} \leq \|C_1^k\| = k.$$

*Note that* $\boldsymbol{C}_2$ *contains also* hyperbolic *elements (namely matrices* $C_I$ *with* $|\,tr\, C_I| > 2$*), e.g.*

$$C_1 C_2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

*so that in the sets* $\boldsymbol{C}_{2,k} = \{C_I, |I| = k\}$ *there are some elements whose norm grows polynomially and some others whose norm grows exponentially with* $k$*, similarly to what happens for the cubic and Apollonian gaskets (see Figs. 1 and 2).*

*This elementary but still non-trivial example was suggested to the author by I.A. Dynnikov and was the starting point for the author's study of the asymptotics of norm's growth in semigroups of linear maps in full generality.*

**Example 4.** *Suppose that* $A_1, \dots, A_m \in M_n(Z)$*, where* $Z = \mathbb{Z}$ *or* $\mathbb{Z}[i]$*, generate* freely $\boldsymbol{A}$. *Then* $\boldsymbol{A}$ *is a gasket, since in* $M_n(Z)$ *there are only finitely many matrices whose norm is smaller than any fixed* $r > 0$ *and by the freedom hypothesis the products of any number of* $A_i$ *are all distinct.*

Note that clearly if $\boldsymbol{A}$ is a gasket then so is also every semigroup conjugated to it, as well as every semigroup obtained from it by multiplying all elements by a constant $\lambda$ such that $|\lambda| > 1$.

**Definition 2.** *Let $\mathcal{A}$ be a $n$-gasket and $M \in GL_n(K)$. We call* zeta function *of $\mathcal{A}_M$ the series*

$$\zeta_{\mathcal{A}_M}(s) = \sum_{I \in \mathcal{I}^m} \frac{1}{\|A_I M\|^s}.$$

*We call* exponent *of $\mathcal{A}_M$ the number $s_{\mathcal{A}_M}$ defined as follows:*

$$s_{\mathcal{A}_M} = \sup_{s \geq 0} \{ s \,|\, \zeta_{\mathcal{A}_M}(s) = \infty \}.$$

Note that, if $s_{\mathcal{A}_M} < \infty$, we also have that $s_{\mathcal{A}_M} = \inf_{s \geq 0}\{s \,|\, \zeta_{\mathcal{A}_M}(s) < \infty\}$.

**Example 5.** *Let $\mathcal{A}$ be the (parabolic) gasket generated by*

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; A_2 = A_1 \in SL_2(\mathbb{N}).$$

*In this case $\|A_{I_k}\| = k$, so that $\zeta_{\mathcal{A}}(s) = \sum_{k \in \mathbb{N}} 2^k k^{-s}$ diverges for all $s$, i.e. $s_{\mathcal{A}} = \infty$.*

*On the contrary, let $\mathcal{B}$ be the (hyperbolic) gasket generated by*

$$B_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \; B_2 = B_1 \in GL_2(\mathbb{N}).$$

*Then $\|B_{I_k}\| = F_{2k+2}$ where $F = (0, 1, 1, 2, 3, 5, \dots)$ is the Fibonacci sequence. Hence asymptotically $\|B_{I_k}\| \simeq g^{2k}$, where $g = \frac{1+\sqrt{5}}{2}$ is the golden ratio, and so $\zeta_{\mathcal{B}}(s)$ diverges or converges with $\sum_{k \in \mathbb{N}} 2^k g^{-2sk}$, i.e. $s_{\mathcal{B}} = \frac{1}{2\log_2 g}$.*

Next proposition shows that norms of matrices in $\mathcal{A}_M$ have the same asymptotic properties of those in $\mathcal{A}$ for every $M \in GL_n(K)$:

**Proposition 1.** *Let $\mathcal{A}$ be a $m$-gasket and $M \in GL_n(K)$. Then $\mathcal{A}_M$ is a $m$-gasket and $s_{\mathcal{A}_M} = s_{\mathcal{A}}$.*

*Proof.* It is a direct consequence of $n\|P\|\|M\| \geq \|PM\| \geq \frac{\|P\|}{n\|M^{-1}\|}$. □

Showing that hyperbolic gaskets have a finite exponent does not require any effort:

**Proposition 2.** *Let $\mathcal{A} : \mathcal{I}^m \to M_n(K)$ be a hyperbolic gasket. Then $s_{\mathcal{A}}$ is finite.*

*Proof.* Since $\mathcal{A}$ is hyperbolic then $\|A_I\| \geq A\alpha^{|I|}$ for some $A > 0$ and $\alpha > 1$. Hence

$$\zeta_{\mathcal{A}}(s) \leq \sum_{k=0}^{\infty} m^k A^s \alpha^{ks} = \sum_{k=0}^{\infty} A^s m^{k(1-s\log_m \alpha)},$$

so that $\zeta_{\mathcal{A}}(s) \leq \infty$ for $s \geq \log_\alpha m$, namely $s_{\mathcal{A}} \leq \log_\alpha m$. $\square$

As Example 5 shows, proving a similar statement for the parabolic case we must require some growth condition on the norms of products.

**Definition 3.** *We say that a gasket $\mathcal{A} : \mathcal{I}^m \to M_n(K)$ is* fast *if there is a constant $c > 0$ such that*

$$\|A_{IJ}\| \geq c\|A_I\|\|A_J\|.$$

*for every multi-indices $I \in \mathcal{I}^m$ and $J \in \mathcal{J}^m \cdot \mathcal{I}^m$. We call*

$$c_{\boldsymbol{A}} = \inf_{\substack{I\in\mathcal{I}^m \\ J\in\mathcal{J}^m\cdot\mathcal{I}^m}} \frac{\|A_{IJ}\|}{\|A_I\|\|A_J\|}$$

*the* coefficient *of the gasket.*

**Example 6.** *Consider the parabolic and hyperbolic gaskets of Example 5. Since*

$$\|A_1^{k'} A_2 A_1^k\| = \|A_1^{k+k'+1}\| = 1 + k + k'$$

*and $\inf_{k,k'\geq 1}\{(1 + k + k')/(kk')\} = 0$, $\mathcal{A}$ is not fast.*

*On the contrary, since any product of $N = k + k'$ copies of $B_{1,2}$ is equal to $B_1^N$ and*

$$\|B_1^{k'} B_1^k\| = \|B_1^{k+k'}\| = F_{2k+2k'} > F_{2k'} F_{2k-2} = \|B_1^{k'}\|\|B_1^{k-1}\|,$$

*then $\mathcal{B}$ is fast with $c_{\mathcal{B}} \geq 1$.*

**Example 7.** *The (parabolic) cubic gasket $\boldsymbol{C}_2$ of Example 3 is fast. Indeed consider first $J = 21L$, with $C_L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that*

$$C_J = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + c & b + d \\ a + 2c & b + 2d \end{pmatrix}.$$

*Clearly* $\|C_J\| = \max\{a+2c, b+2d\} \leq 2\max\{a+c, b+d\}$ *and therefore*

$$\|MC_J\| \geq \frac{1}{2}\|M\|\|C_J\|$$

*for every* $M \in SL_2(\mathbb{N})$. *The same argument applies to* $J = 12L$. *Since* $\|C_1^{k'}C_1C_2C_1^k\| = k'(k+1)$, $\|C_1^{k'}\| = k'$ *and* $\|C_1C_2C_1^k\| = 2k+1$ *it follows at once that in fact* $c_{\boldsymbol{C}_2} = 1/2$.

**Example 8.** *A hyperbolic gasket is not necessarily fast. Consider for instance* $\mathcal{A} : \mathcal{I}^2 \to GL_2(\mathbb{Z}[i])$ *with*

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & \alpha \end{pmatrix}, A_2 = \begin{pmatrix} \beta & 0 \\ 0 & 2 \end{pmatrix},$$

*where* $|\alpha|, |\beta| = 1$. $\mathcal{A}$ *is hyperbolic since every* $A_I$, $|I| = 2k$, *contains an entry with modulus* $2^{k'}$ *and* $k' \geq k$. *On the other side* $\mathcal{A}$ *is not fast. Indeed* $\|A_1^kA_1A_2^{k'}\| = 2^{-k}\|A_1^k\|\|A_1A_2^{k'}\|$ *for all* $k \leq k'$ *and so*

$$\inf_{I\in\mathcal{I}^2, J\in\mathcal{J}^2} \frac{\|A_{IJ}\|}{\|A_I\|\|A_J\|} = 0.$$

**Proposition 3.** *If* $\mathcal{A}$ *is a fast gasket with coefficient* $c$, *the gasket* $\lambda\mathcal{A}$ *is a fast gasket for every* $\lambda \neq 0$ *with coefficient* $c/|\lambda|$.

*Proof.* $\|\lambda A_{IJ}\| = |\lambda|\|A_{IJ}\| \geq c|\lambda|\|A_I\|\|A_{J'}\| = \frac{c}{|\lambda|}\|\lambda A_I\|\|\lambda A_J\|$ ☐

Our main *algebraic* results on the exponent of a gasket are the following:

**Theorem 1** (Exponent of a fast gasket)**.** *If* $\mathcal{A} : \mathcal{I}^m \to M_n(K)$ *is a fast gasket then* $0 < s_{\mathcal{A}} < \infty$.

In the more detailed discussion in Section 3.2.2 we also show how to build explicitly two sequences of monotonically decreasing functions $f_{\mathcal{A},k}(s)$ and $g_{\mathcal{A},k}(s)$ such that, for all $k$ from some $\bar{k}$ on, $s_{\mathcal{A}} \in [g^{-1}_{\mathcal{A},k}(1), f^{-1}_{\mathcal{A},k}(1)]$ and $|f^{-1}_{\mathcal{A},k}(1) - g^{-1}_{\mathcal{A},k}(1)| \leq a\log k$ for some $a > 0$. These sequences will be used in Section 4 to evaluate analytical bounds for the exponents of a few semigroups.

**Theorem 2** (Alternate characterization of the exponent of a gasket)**.** *Let* $\mathcal{A} : \mathcal{I}^m \to M_n(K)$ *be a gasket with* $s_{\mathcal{A}} < \infty$. *Then the function*

$$\xi_{\mathcal{A}}(s) = \lim_{k\to\infty}\left[\sum_{|I|=k}\|A_I\|^{-s}\right]^{\frac{1}{k}}$$

*is a well defined log-convex positive strictly decreasing function and satisfies the following properties:*

1. $\xi_{\mathcal{A}}(0) = m$;
2. $\xi_{\mathcal{A}}(s) > 1$ *for* $s < s_{\mathcal{A}}$;
3. $\xi_{\mathcal{A}}(s_{\mathcal{A}}) = 1$;
4. $\xi_{\mathcal{A}}(s) < 1$ *for* $s > s_{\mathcal{A}}$ *if* $\mathcal{A}$ *is a hyperbolic gasket;*
5. $\xi_{\mathcal{A}}(s) = 1$ *for* $s > s_{\mathcal{A}}$ *if* $\mathcal{A}$ *is a fast parabolic gasket.*

**Theorem 3** (Exponential growth of norms)**.** *Let* $\mathcal{A} : \mathcal{I}^m \to M_n(K)$ *be a gasket with* $s_{\mathcal{A}} < \infty$. *Then*

$$\lim_{k\to\infty} \frac{\log N_{\mathcal{A}_M}(k)}{\log k} = s_{\mathcal{A}}$$

*for every* $M \in GL_n(K)$.

It is natural to ask whether this result can be made stronger, namely whether $N_{\mathcal{A}}(k) \asymp k^{s_{\mathcal{A}}}$. This leads to the following:

**Open Question 1.** *Based on numerical investigations, Boyd [Boy82] conjectured that, in case of the Apollonian gasket, the number* $N(k)$ *of circles whose curvature is not larger than* $k$ *(which, in our setting, corresponds to the number of Hirst matrices whose norm is not larger than* $k$*) were only* weakly *asymptotic to* $k^s$, *where* $s$ *is the dimension of the Apollonian gasket, namely that*

$$N(k) \simeq Ak^s \log^t(k/B)$$

*for some* $A, B, t > 0$. *Recently Kontorovich and Oh [KO11] disproved this conjecture by showing that actually* $N(k)$ *is* strongly *asymptotic to* $k^s$, *namely* $N(k) \asymp k^s$ *for all* $k \in \mathbb{N}$.

*We pose the following question: is* $N_{\mathcal{A}}(k)$ strongly *asymptotic to* $k^{s_{\mathcal{A}}}$ *for every gasket* $\mathcal{A}$ *with a finite exponent? Or is there some extra condition that must be put on* $\mathcal{A}$ *to ensure this behaviour?*

**Hausdorff dimension of limit sets of discrete subsemigroups of real and complex projective automorphisms.** Recall that the projective space $KP^{n-1}$ is defined as the set of all 1-dimensional linear subspaces of

$K^n$. We denote by $[v]$ the 1-dimensional subspace of $K$ containing $v$, so that $[\cdot] : K^n \to KP^{n-1}$ is a projection with kernel $K \setminus \{0\}$. Since linear maps preserve linear subspaces, every automorphism $f : K^n \to K^n$ induces a projective automorphism $\psi_f : PK^{n-1} \to PK^{n-1}$ defined by $\psi_f([v]) = [f(v)]$.

We endow $PK^{n-1}$ with the *round* metric tensor, namely the metric with constant curvature equal to 1, and with the distance and measure induced by it. In every compact of every chart of $KP^{n-1}$ this distance is Lipschitz equivalent to the Euclidean distance in the chart and the measure is in the same measure class of the Lebesgue measure in the chart, so that we can forget about the definition and work directly with the Euclidean distance and Lebesgue measure in any chart.

In this paper by *iterated function system* (IFS) on a metric space $(X, d)$ we mean a homomorphism $F : \mathcal{I}^m \to C(X)$, where $C(X)$ is the set of continuous maps from $X$ into itself, such that the generators $f_i \stackrel{\text{def}}{=} F(i)$, $i = 1, \ldots, m$, are non-expansive maps, namely there exist $K_i \in (0, 1]$ such that $d(f_i(x), f_i(y)) \leq K_i d(x, y)$ for all $x, y \in X$. If $K_i < 1$ for all $i$ then we say that the IFS is *hyperbolic*, otherwise that is *parabolic*. Recall that to a hyperbolic IFS $F$ on $X$ it corresponds a unique compact set $R_F \subset X$, which is also equal to the set of limit points of the orbit of almost every point $x \in X$ under the action of $F$, whose Hausdorff dimension, which we denote by $\dim_H R_F$, is often non-integer and can be evaluated through some property of the $f_i$, e.g. their coefficients $K_i$ or, if regular enough, their derivatives (e.g. see Chapter 9 of [Fal90] for more details and examples). Finally, recall that the $f_i$ satisfy the *open set* condition if there exists a relatively compact subset $V \subset X$ such that $V \supset \sqcup_{i=1}^m f_i(V)$. Note that this condition in particular implies that there is no relation between the $f_i$.

In a seminal paper D. Sullivan [Sul84] related the Hausdorff dimension of the limit set $R_\Gamma \subset \mathbb{C}P^1$ of a (geometrically finite) Kleinian (i.e. discrete) subgroup $\Gamma \subset PSL_2(\mathbb{C})$ to a critical exponent of $\Gamma$ defined in the context of hyperbolic geometry (the recent results by Kontorovich and Oh quoted in Open Question 1 are in fact based on this result). In the same spirit here we relate (partly in form of conjecture) the exponent $s_{\boldsymbol{A}}$ of a sub*semi*group of $SL_n^\pm(\mathbb{R})$ or $SL_n(\mathbb{C})$ to the Hausdorff or Box dimensions of the limit set $R_{\boldsymbol{A}} \subset KP^{n-1}$, $K = \mathbb{R}$ or $\mathbb{C}$, of the subsemigroup of $PSL_n^\pm(\mathbb{R})$ or $PSL_n(\mathbb{C})$ naturally induced by $\boldsymbol{A}$. This somehow complements recent results on the geometry of residual sets of real projective IFS by Barnsley and Vince [BV10] and on complex projective IFS by Vince [Vin12].

We denote by $f_1, \dots, f_m$ linear volume-preserving automorphisms of $K^n$, by $A_i \in SL^{\pm}_n(K)$ the matrices representing the $f_i$ in some coordinate system, by $\boldsymbol{A}$ the semigroup generated by the $A_i$ and by $\psi_i \in PSL^{\pm}_n(K)$ the projective automorphism naturally induced by $f_i$.

Our main *geometric* results on the exponent of a gasket are the following:

**Theorem 4** (Semigroups of $PSL^{\pm}_2(\mathbb{R})$)**.** *Let $f_1, \dots, f_m : \mathbb{R}^2 \to \mathbb{R}^2$ be such that the induced maps $\psi_i \in PSL^{\pm}_2(\mathbb{R})$ satisfy the open set condition with respect to some proper subset $V \subset \mathbb{R}P^1$ and that, in some affine chart $\varphi : \mathbb{R}P^1 \to \mathbb{R}$, the $\psi_i$ are contractions on $\varphi(\overline{V})$ with respect to the Euclidean distance. Let $R_{\boldsymbol{A}} = \cap_{k=1}^{\infty} \left(\cup_{|I|=k} \psi_I(V)\right)$ be the corresponding residual set. Then $2 \dim_H R_{\boldsymbol{A}} = s_{\boldsymbol{A}}$.*

The simplicity of the geometry of $\mathbb{R}P^1$ suggests that the theorem above can be extended to more general families of $f_i$, leading to the following:

**Open Question 2.** *Let $\boldsymbol{A} \subset SL^{\pm}_2(\mathbb{R})$ be a free finitely generated semigroup with finite exponent $s_{\boldsymbol{A}}$ and $\Psi_{\mathcal{A}}$ the corresponding free discrete subsemigroup of $PSL^{\pm}_2(\mathbb{R})$. Under which general assumptions is the Hausdorff dimension of the limit set of $\Psi_{\boldsymbol{A}}$ related to $s_{\boldsymbol{A}}$?*

An identical theorem holds in the complex case:

**Theorem 5** (Semigroups of $PSL_2(\mathbb{C})$)**.** *Let $f_1, \dots, f_m : \mathbb{C}^2 \to \mathbb{C}^2$ be such that the induced maps $\psi_i \in PSL_2(\mathbb{C})$ satisfy the open set condition with respect to some proper subset $V \subset \mathbb{C}P^1$ and that, in some affine chart $\varphi : \mathbb{C}P^1 \to \mathbb{R}$, the $\psi_i$ are contractions on $\varphi(\overline{V})$ with respect to the Euclidean distance. Let $R_{\boldsymbol{A}} = \cap_{k=1}^{\infty} \left(\cup_{|I|=k} \psi_I(V)\right)$ be the corresponding residual set. Then $2 \dim_H R_{\boldsymbol{A}} = s_{\boldsymbol{A}}$.*

Even in this case there is evidence that the claim of the theorem can be generalized further, at least to some parabolic case, namely when the $\psi_i$ are not contractive but just non-expanding. This is, for example, the case for the Apollonian gasket as shown in the introduction.

For $n > 2$ things get geometrically much more complicated for both $\mathbb{R}$ and $\mathbb{C}$ and we cannot claim any general result. In order to provide motivations for further studies of this case and as a source of examples of non-trivial fast gaskets, in Section 4 we introduce for free semigroups $\boldsymbol{A} \subset SL^{\pm}_n(\mathbb{R})$ the notion of *real* and *complex self-projective Sierpinski gaskets*, the name being due to the fact that to such $\boldsymbol{A}$'s it correspond the same "cut-out" construction of

the standard Sierpinski gasket and of its multi-dimensional generalizations. In particular to each such gaskets it corresponds a curvilinear simplex $T_{\boldsymbol{A}} \subset \mathbb{R}P^{n-1}$ invariant by the action induced on it by $\boldsymbol{A}$ and a compact invariant set $R_{\boldsymbol{A}} \subset T_{\boldsymbol{A}}$ obtained by subtracting recursively curvilinear polyhedra from $T_{\boldsymbol{A}}$.

In particular in Section 4.3 we show that in $PSL^{\pm}_n(\mathbb{R})$, $n \geq 3$, there exist a smooth 1-parameter family of fast free semigroups $\boldsymbol{A}_t$, $t \in [1, \infty)$, such that: 1) $\boldsymbol{A}_t$ is a hyperbolic IFS for $t \in (1, 4)$; 2) the extremal gasket $\boldsymbol{A}_1$ is the (parabolic, multi-dimensional generalization of the) cubic gasket; 3) $\boldsymbol{A}_2$ is the (multi-dimensional real self-projective generalization of the) standard Sierpinski gasket. Siilarly, in Section 4.2 we correspondingly show that in $PSL_2(\mathbb{C})$ there exist a smooth 1-parameter family of semigroups $\boldsymbol{A}_t$, $t \in [1/5, \alpha]$, where $\alpha \simeq 0.651$, such that: 1) $\boldsymbol{A}_t$ is a hyperbolic IFS for $t \in (1/5, \alpha)$; 2) the extremal semigroup $\boldsymbol{A}_{1/5}$ is the (parabolic) Apollonian gasket and $\boldsymbol{A}_{1/2}$ is the (complex self-projective generalization of the) standard Sierpinski gasket. This shows that the cubic and Apollonian gaskets are both natural parabolic deformations of the standard Sierpinski gasket, the first in the real setting and the second in the complex setting.

Finally, based on numerical and analytical results in Section 4.3, we claim the following about the box dimension of the residual set:

**Conjecture 1.** *Let $\boldsymbol{A} \subset SL^{\pm}_n(\mathbb{R})$ be a real projective Sierpinski gasket and $R_{\boldsymbol{A}}$ its residual set. Then, under suitable natural assumptions,*

$$(n+1)\dim_B R_{\boldsymbol{A}} \geq n s_{\boldsymbol{A}}.$$

# 3 Asymptotic growth of norms

## 3.1 The case $m = 1$.

Let us briefly discuss the exponent of a gasket in the trivial case $m = 1$, since these results will be useful later in this section.

Here $\mathcal{I} = \mathbb{N}$, $\mathcal{A}$ is generated by a single matrix $A \in M_n(K)$, $K = \mathbb{R}$ or $\mathbb{C}$, and $\mathcal{A}(k) = A^k$. In order for $\mathcal{A}$ to be a gasket then it is necessary and sufficient that $A$ has an eigenvalue with modulus larger than 1 (in which case it will be a hyperbolic gasket) or that has a single eigenvalue of modulus 1 and its spectral norm larger than 1 (in which case it will be a parabolic gasket).

If $\mathcal{A}$ is hyperbolic then $\|A^k\|$ grows exponentially with $k$ and so $s = 0$ is the only exponent that can make the series $\zeta_{\mathcal{A}}(s) = \sum_{k\in\mathbb{N}} \|A^k\|^{-s}$ divergent, i.e. $s_{\mathcal{A}} = 0$.

If $\mathcal{A}$ is parabolic then its generator $A$ has only eigenvalues of modulus 1 and therefore its norm grows as some polynomial of degree $d < n$. Hence $\zeta_{\mathcal{A}}(s)$ diverges for $s \leq 1/d$ and is finite for $s > 1/d$, i.e. $s_{\mathcal{A}} = 1/d$.

Now consider the number $N_{\mathcal{A}}(r)$ of powers of $A$ whose norm is not larger than $r$. When $\|A^k\|$ is a polynomial of order $d$ their number grows as $r^{1/d}$, so that the limit $\lim_{k\to\infty} \log N_{\mathcal{A}}(r)/\log r = 1/d$ equals $s_{\mathcal{A}}$. When $\|A^k\|$ grows exponentially then $N_{\mathcal{A}}(r)$ grows logarithmically so that, again, $\lim_{k\to\infty} \log N_{\mathcal{A}}(r)/\log r = 0$ equals $s_{\mathcal{A}}$. Incidentally, this proves Theorems 1 and 3 for $m = 1$.

## 3.2 The case $m > 1$.

When there is more than one generator things are qualitatively different: the number of terms of order $k$ (i.e. products of $k$ generators) increases exponentially and there can be coexistence of polynomial and exponential growths of norms for terms of the same order. E.g. we already pointed out this behaviour for the $\boldsymbol{C}_2$ semigroup in Example 7 and the same happens for all semigroups $\boldsymbol{C}_n$ (see Section 4.3.2).

Following Boyd's arguments in [Boy73b] we are able to find upper and lower bounds for $\zeta_{\mathcal{A}_M}(s)$ using only the series of the norms of the "diagonal" terms $A_D M$, $D \in \mathcal{D}^m$, and of those "next-to-diagonal" ones $A_J M$, $J \in \mathcal{J}^m$. The key point of the next arguments is the following elementary recursive re-writing of the zeta function:

$$\zeta_{\mathcal{A}_M}(s) = \sum_{D\in\mathcal{D}^m} \frac{1}{\|A_D M\|^s} + \sum_{J\in\mathcal{J}^m} \zeta_{\mathcal{A}_{A_J M}}(s). \tag{2}$$

### 3.2.1 A fundamental set of inequalities

Using (2) we can now write the following fundamental inequality:

**Proposition 4.** *Let $\mathcal{A} : \mathcal{I}^m \to M_n(K)$ be a fast gasket with coefficient $c_{\mathcal{A}}$*

*and let $J \in \mathcal{J}^m \cdot \mathcal{I}^m$ and $I \in \mathcal{I}^m$. Then the following inequalities hold:*

$$n^{-s}\|A_J\|^{-s}\zeta_{\mathcal{A}}(s) \le \zeta_{\mathcal{A}_{A_J}}(s) \le c_{\mathcal{A}}^{-s}\|A_J\|^{-s}\zeta_{\mathcal{A}}(s) \tag{3}$$

$$\zeta_{\mathcal{A}_{A_I}}(s) \ge \nu_{\mathcal{A}_{A_I}}(s) + n^{-s}\mu_{\mathcal{A}_{A_I}}(s)\zeta_{\mathcal{A}}(s) \tag{4}$$

$$\zeta_{\mathcal{A}_{A_I}}(s) \le \nu_{\mathcal{A}_{A_I}}(s) + c_{\mathcal{A}}^{-s}\mu_{\mathcal{A}_{A_I}}(s)\zeta_{\mathcal{A}}(s), \tag{5}$$

*where* $\nu_{\mathcal{A}_M}(s) = \sum_{D\in\mathcal{D}^m} \frac{1}{\|A_D M\|^s}$ *and* $\mu_{\mathcal{A}_M}(s) = \sum_{J\in\mathcal{J}^m} \frac{1}{\|A_J M\|^s}$.

*Proof.* The left side of (3) is a direct consequence of (1), its right side of the definition of fast gasket. The starting point to prove inequalities (4,5) is (2), from which we get

$$\zeta_{\mathcal{A}_{A_I}}(s) = \nu_{\mathcal{A}_{A_I}}(s) + \sum_{J\in\mathcal{J}^m} \zeta_{\mathcal{A}_{A_{JI}}}(s).$$

Applying the right side of (3) to the summation above we get that

$$\sum_{J\in\mathcal{J}^m} \zeta_{\mathcal{A}_{A_{JI}}}(s) \le \sum_{J\in\mathcal{J}^m} \frac{1}{c_{\mathcal{A}}^s\|A_{JI}\|^s}\zeta_{\mathcal{A}}(s) = \frac{1}{c_{\mathcal{A}}^s}\mu_{\mathcal{A}_{A_I}}(s)\zeta_{\mathcal{A}}(s).$$

Hence (5) follows and analogously it is proven (4). □

**Remark 1.** *The function $\nu_{\mathcal{A}_M}$ has the same complexity of the zeta functions of* 1*-gaskets. Indeed let $\mathcal{A}_i = \langle A_i\rangle$ be the 1-gaskets generated by the $m$ generators of $\mathcal{A}$. Then $\nu_{\mathcal{A}_M}(s) = \sum_{1\le i\le m}\zeta_{\mathcal{A}_i}(s)$. The ultimate idea of this section is to exploit this fact to find upper and lower bounds for $\zeta_{\mathcal{A}}$ using the much simpler $\nu_{\mathcal{A}}$.*

**Remark 2.** *For finite $s$, $\mu_{\mathcal{A}}$ and $\nu_{\mathcal{A}}$, converge or diverge together. Indeed if $J = iD \in \mathcal{J}^m$ then $J' = D \in \mathcal{D}^m$ and*

$$\frac{\|A_D\|}{n\|A_i^{-1}\|} \le \|A_{iD}\| \le n\|A_i\|\|A_D\|,$$

*so that*

$$(m-1)n^{-s}\min_{1\le i\le m}\|A_i\|^{-s}\nu_{\mathcal{A}}(s) \le \mu_{\mathcal{A}}(s) \le (m-1)n^s\max_{1\le i\le m}\|A_i^{-1}\|^s\nu_{\mathcal{A}}(s).$$

*The same is true for $s\to\infty$ after some finite number of terms (those of "too small" norm) is removed from the two series. For similar reasons $\nu_{\mathcal{A}_M}$ and $\mu_{\mathcal{A}_M}$ converge or diverge together with $\nu_{\mathcal{A}}$ for every $M \in GL_n(K)$.*

Let us now examine closely the two inequalities (4,5) for $I = 0$. The first one becomes

$$\zeta_{\mathcal{A}}(s) \geq \nu_{\mathcal{A}}(s) + n^{-s}\mu_{\mathcal{A}}(s)\zeta_{\mathcal{A}}(s) \tag{6}$$

Since we are going to use this inequality to get lower bounds for $\zeta_{\mathcal{A}}(s)$, we can proceed without loss of generality by assuming that $\zeta_{\mathcal{A}}(s) < \infty$. Then we get that, for $n^{-s}\mu_{\mathcal{A}}(s) < 1$,

$$\zeta_{\mathcal{A}}(s) \geq \frac{\nu_{\mathcal{A}}(s)}{1 - n^{-s}\mu_{\mathcal{A}}(s)}. \tag{7}$$

Analogously the right one becomes

$$\zeta_{\mathcal{A}}(s) \leq \nu_{\mathcal{A}}(s) + c_{\mathcal{A}}^{-s}\mu_{\mathcal{A}}(s)\zeta_{\mathcal{A}}(s). \tag{8}$$

This time though, since we are going to use this inequality to provide upper bounds to the zeta function, we need to truncate the infinite series to a finite sum to ensure we are dealing with finite numbers. A natural recursive definition, inspired by the structure of (2), is

$$\begin{array}{rcl} \zeta^0_{\mathcal{A}_M}(s) & = & \nu^0_{\mathcal{A}_M}(s) \\ \zeta^\ell_{\mathcal{A}_M}(s) & = & \nu^\ell_{\mathcal{A}_M}(s) + \displaystyle\sum_{J \in \mathcal{J}^m_{\ell+1}} \zeta^{\ell+1-|J|}_{\mathcal{A}_{A_J M}}(s), \ \ell \geq 1 \end{array} ,$$

where $\nu^\ell_{\mathcal{A}_M}$ is the necklace sum truncated at the order $\ell$.

**Proposition 5.** *Consider the sets $\mathcal{P}^m_\ell \subset \mathcal{I}^m$ recursively defined as*

$$\begin{array}{rcl} \mathcal{P}^m_0 & = & \mathcal{D}^m_0, \\ \mathcal{P}^m_\ell & = & \mathcal{D}^m_\ell \bigcup \left[ \displaystyle\bigcup_{J \in \mathcal{J}^m_{\ell+1}} \mathcal{P}^m_{\ell+1-|J|} \cdot J \right], \ \ell \geq 1. \end{array}$$

*Then the following properties hold:*

1. $\mathcal{P}^m_\ell \subset \mathcal{P}^m_{\ell+1}$ *for all* $\ell \geq 0$*;*
2. $\cup_{\ell \geq 0} \mathcal{P}^m_\ell = \mathcal{I}^m$*;*
3. $\zeta^\ell_{\mathcal{A}_M}(s) = \sum_{I \in \mathcal{P}^m_\ell} \|A_I M\|^{-s}$.

*Proof.* 1. We prove this by induction assuming that $\mathcal{P}^m_k \subset \mathcal{P}^m_{k+1}$ for all $k \le \ell - 1$. Now, if $I \in \mathcal{P}^m_\ell$ then either $I \in \mathcal{D}^m_\ell$ or $I = PJ$ with $J \in \mathcal{J}^m_{\ell+1}$ and $P \in \mathcal{P}^m_{\ell+1-|J|}$. In the first case $I \in \mathcal{P}^m_{\ell+1}$ since $\mathcal{D}^m_\ell \subset \mathcal{D}^m_{\ell+1}$. In the second case we have that $\ell + 1 - |J| \le \ell - 1$ since every element of $\mathcal{J}^m$ has at least rank two and therefore, by the inductive hypothesis, $\mathcal{P}^m_{\ell+1-|J|} \subset \mathcal{P}^m_{\ell+1-|J|+1}$. Hence, by definition, $I \in \mathcal{P}^m_{\ell+2-|J|} \cdot J \subset \mathcal{P}^m_{\ell+1}$.

2. Notice first of all that every index $I \in \mathcal{I}^m$ either belongs to $\mathcal{D}^m$ (and so to some $\mathcal{P}^m_\ell$) or it can be factored out as a product $I = J_1 \cdots J_k$ with $J_i \in \mathcal{J}^m$. This factorization simply consists in singling out the patterns of the form $i_1 \ne i_2 = i_3 = \cdots = i_p$ inside $I$ and is clearly unique. Let $K = \max_{1\le i\le k} |J_i|$. By construction $J_i \in \mathcal{P}^m_K$ for every $1 \le i \le k$. Then $\mathcal{P}^m_{2K+1} \supset \bigcup_{J\in\mathcal{J}^m_{2K+2}} \mathcal{P}^m_K \cdot J$ contains all products $J_iJ_j$, $1 \le i, j \le k$ and similarly $\mathcal{P}^m_{kK+1}$ contains all possible products of $k$ of the $J_i$, namely $I \in \mathcal{P}^m_{kK+1}$.

3. Let us write $\zeta^\ell_{\mathcal{A}_M}(s) = \sum_{I\in\mathcal{G}^\ell} \|A_I M\|^{-s}$. Then if $I \in \mathcal{G}^\ell$ either $\|A_I M\|^{-s}$ appears in $\nu^\ell_{\mathcal{A}_M}(s)$, in which case $I \in \mathcal{D}^m_\ell$ by definition, or in $\zeta^{\ell+1-|J|}_{\mathcal{A}_{A_J M}}(s)$, in which case $I = KJ$ with $K \in \mathcal{G}^{\ell+1-|J|}$. This is exactly the rule that defines recursively the $\mathcal{P}^m_\ell$. Since we also have by definition of $\zeta^0_{\mathcal{A}_M}(s)$ that $\mathcal{G}^0 = \mathcal{D}^m_0$ it follows that $\mathcal{G}^\ell = \mathcal{P}^m_\ell$. □

**Corollary 1.** *Let $\mathcal{A}$ be a fast gasket. Then the $\zeta^\ell_{\mathcal{A}_M}(s)$ satisfy the following properties:*

$$\zeta^\ell_{\mathcal{A}_M}(s) \le \zeta^{\ell+1}_{\mathcal{A}_M}(s) \text{ for all } \ell \ge 0; \tag{9}$$

$$\lim_{\ell\to\infty} \zeta^\ell_{\mathcal{A}_M}(s) = \zeta_{\mathcal{A}_M}(s); \tag{10}$$

$$n^{-s}\|A_J\|^{-s}\zeta^\ell_{\mathcal{A}}(s) \le \zeta^\ell_{\mathcal{A}_{A_J}}(s) \le c^{-s}_{\mathcal{A}}\|A_J\|^{-s}\zeta^\ell_{\mathcal{A}}(s); \tag{11}$$

$$\zeta^\ell_{\mathcal{A}_{A_I}}(s) \le \nu_{\mathcal{A}_{A_I}}(s) + c^{-s}_{\mathcal{A}}\mu_{\mathcal{A}_{A_I}}(s)\zeta^\ell_{\mathcal{A}}(s). \tag{12}$$

*Proof.* (9,10) are a direct consequence of points 1. and 2. of the previous proposition. (11) is a direct consequence of (1) (left) and of the definition of fast gasket (right). Using the rhs of (11) and then (9) we get that

$$\zeta^\ell_{\mathcal{A}_M}(s) \le \nu^\ell_{\mathcal{A}_M}(s) + c^{-s} \sum_{J\in\mathcal{J}^m_{\ell+1}} \|A_J M\|^{-s}\zeta^{\ell+1-|J|}_{\mathcal{A}}(s) \le \nu^\ell_{\mathcal{A}_M}(s) + c^{-s}\mu^{\ell+1}_{\mathcal{A}_M}(s)\zeta^\ell_{\mathcal{A}}(s)$$

from which (12) follows. □

Using the monotonicity in $\ell$ of $\nu^\ell_{\mathcal{A}_M}(s)$ and $\mu^\ell_{\mathcal{A}_M}(s)$ and setting $M = \mathbb{1}_n$ we get that, in particular,

$$\zeta^\ell_{\mathcal{A}}(s) \le \nu_{\mathcal{A}}(s) + c^{-s}_{\mathcal{A}}\mu_{\mathcal{A}}(s)\zeta^\ell_{\mathcal{A}}(s).$$

From this we deduce that, when $\mu_{\mathcal{A}}(s) < c^s_{\mathcal{A}}$, $\zeta^\ell_{\mathcal{A}}(s) \leq \frac{\nu_{\mathcal{A}}(s)}{1-c_{\mathcal{A}}^{-s}\mu_{\mathcal{A}}(s)}$ for all $\ell$ and therefore, finally,

$$\zeta_{\mathcal{A}}(s) \leq \frac{\nu_{\mathcal{A}}(s)}{1-c_{\mathcal{A}}^{-s}\mu_{\mathcal{A}}(s)}. \tag{13}$$

In the next section we will use these bounds to build bounds for $s_{\mathcal{A}}$.

### 3.2.2 The exponent of a fast gasket

The idea of next theorem comes from the following observation, valid in the particular case when all generators $A_i$ of $\mathcal{A}$ have non-negative coefficients and norms larger than the maximum between 1 and the inverse of the coefficient $c_{\mathcal{A}}$ of the gasket.

Consider (7). The function $g_{\mathcal{A}}(s) = n^{-s}\mu_{\mathcal{A}}(s)$ is clearly monotonically decreasing with $s$, defined on $(0,\infty)$ and $g(0,\infty) = (0,\infty)$. In particular the set $g_{\mathcal{A}}(s) < 1$ is not empty and therefore (7) holds for $s > s_g$, with $s_g = g_{\mathcal{A}}^{-1}(1)$. When we let $s \to s_g^+$ the right hand side of (7) goes to infinity so that $\zeta_{\mathcal{A}}(s_g)$ diverges, namely $s_{\mathcal{A}} \geq s_g$.

Now consider (13). The function $f_{\mathcal{A}}(s) = c^{-s}\mu_{\mathcal{A}}(s)$ satisfies the same properties listed above for $g_{\mathcal{A}}$. Let $s_f = f_{\mathcal{A}}^{-1}(1)$. Then $f_{\mathcal{A}}(s) < 1$ for $s > s_f$ so that (7) holds. This proves that $\zeta_{\mathcal{A}}(s) < \infty$ for all $s > s_f$, namely $s_f \geq s_{\mathcal{A}}$.

This simple argument not only grants us that $s_{\mathcal{A}}$ is finite but also provides for it non-trivial lower and upper bounds. In order to obtain a similar result in full generality we need to refine (13). This will lead us to refine also (7) and to generate a pair of sequences converging to $s_{\mathcal{A}}$ from the left and from the right at logarithmic speed. The idea is to apply over and over recursively first the inequalities (4,5) and then the inequality (3) to the truncated zeta function.

The starting point is the sequence of sets of multi-indices $\mathcal{Q}_{\mathcal{A},k}$ built as follows. We define $\mathcal{Q}^0_{\mathcal{A},k} = \mathcal{J}^m$. Then we consider the sets recursively defined as

$$\mathcal{Q}^\ell_{\mathcal{A},k} = \left\{J \in \mathcal{Q}^{\ell-1}_{\mathcal{A},k} \,|\, \|A_J\| > k\right\} \bigcup \mathcal{J}^m \cdot \left\{J \in \mathcal{J}^m \,|\, J \in \mathcal{Q}^{\ell-1}_{\mathcal{A},k}, \|A_J\| \leq k\right\}$$

with $\ell \geq 1$.

**Proposition 6.** *For every gasket $\mathcal{A}$ and every $k > 0$ there exists a $\bar\ell$ such that $\mathcal{Q}^{\bar\ell}_{\mathcal{A},k} = \mathcal{Q}^{\bar\ell+1}_{\mathcal{A},k}$.*

*Proof.* The only thing that the recursive algorithm does is replacing the indices of $\mathcal{Q}^{\ell}_{\mathcal{A},k}$ corresponding to matrices with norm not larger than $k$ with indices of higher order. The set $\mathcal{Q}^{\ell+1}_{\mathcal{A},k}$ thus obtained can still contain indices corresponding to matrices with norm not larger than $k$ but in a finite number of steps all such indices will disappear because, by definition, there is only a finite amount of them. Hence there exists a finite $\bar{\ell}$ such that $\|A_I\| > k$ for all $I \in \mathcal{Q}^{\bar{\ell}}_{\mathcal{A},k}$. Then the algorithm leaves $\mathcal{Q}^{\bar{\ell}}_{\mathcal{A},k}$ unchanged. □

**Definition 4.** *We use the notation $\mathcal{Q}_{\mathcal{A},k} = \mathcal{Q}^{\bar{\ell}}_{\mathcal{A},k}$ and, correspondingly,*

$$f^{\ell}_{\mathcal{A},k}(s) = c_{\mathcal{A}}^{-s} \sum_{J \in \mathcal{Q}^{\ell}_{\mathcal{A},k}} \|A_J\|^{-s}, \qquad f_{\mathcal{A},k}(s) = c_{\mathcal{A}}^{-s} \sum_{J \in \mathcal{Q}_{\mathcal{A},k}} \|A_J\|^{-s} \tag{14}$$

$$g^{\ell}_{\mathcal{A},k}(s) = n^{-s} \sum_{J \in \mathcal{Q}^{\ell}_{\mathcal{A},k}} \|A_J\|^{-s}, \qquad g_{\mathcal{A},k}(s) = n^{-s} \sum_{J \in \mathcal{Q}_{\mathcal{A},k}} \|A_J\|^{-s}. \tag{15}$$

Note that $f_{\mathcal{A},0}(s) = f_{\mathcal{A}}(s)$ and $g_{\mathcal{A},0}(s) = g_{\mathcal{A}}(s)$. Next proposition though shows that, for $k$ big enough, $f_{\mathcal{A},k}$ and $g_{\mathcal{A},k}$ have a nicer behaviour than $f_{\mathcal{A}}$ and $g_{\mathcal{A}}$ do for a general $\mathcal{A}$.

**Proposition 7.** *For every fast gasket $\mathcal{A}$ there exists a $\bar{k}$ and a $\gamma_{\mathcal{A},k} < \infty$ such that $f_{\mathcal{A},k}$ and $g_{\mathcal{A},k}$ are strictly decreasing continuous functions of $s$ defined in $(\gamma_{\mathcal{A},k}, \infty)$ and with image $(0, \infty)$ for all $k > \bar{k}$.*

*Proof.* By construction $f_{\mathcal{A},k}(s)$ and $g_{\mathcal{A},k}(s)$ are proportional to each other and, respectively with constants $c_{\mathcal{A}}^{-s}$ and $n^{-s}$, to the sum of a finite number of functions $\mu^{(k)}_{\mathcal{A}_{A_I}}$ defined as the series $\mu_{\mathcal{A}_{A_I}}$ from which all terms with norm smaller than $k$ have been subtracted.

By Remarks 1 and 2 then

$$n^{-s} \min_{1 \le i \le m} \|A_i\|^{-s} \nu^{(k)}_{\mathcal{A}_{A_I}}(s) \le \mu^{(k)}_{\mathcal{A}_{A_I}}(s) \le (m-1) n^s \max_{1 \le i \le m} \|A_i^{-1}\|^s \sum_{I \in \mathcal{G}} \nu^{(k)}_{\mathcal{A}_{A_I}}(s),$$

where $\mathcal{G} \subset \mathcal{I}^m$ is some finite set of indices and the series $\nu^{(k)}_{\mathcal{A}_{A_I}}$ is equal to $\nu_{\mathcal{A}_{A_I}}$ minus those terms with indices $D \in \mathcal{D}^m$ such that $\|A_{JI}\| \le k$ for all $J \in \mathcal{J}^m$ with $J' = D$. In particular then also $f_{\mathcal{A},k}$ and $g_{\mathcal{A},k}(s)$ are bounded by a finite sum of series $\nu^{(k)}_{\mathcal{A}_{A_I}}(s)$ and so, by the case $m = 1$ of Theorem 1 discussed in Section 3.1, they are finite on some connected non-empty interval $(\gamma_{\mathcal{A},k}, \infty)$.

Now let $\bar{k} = n^2 \max_{1\leq i\leq m} \|A_i^{-1}\| \max_{1\leq i\leq m} \|A_i\|$. For every $k > \bar{k}$ we have that, if the index $DI$ appears in the series $\nu^{(k)}_{\mathcal{A}_{A_I}}$ and $J$ is one of the indices such that $\|A_{JI}\| > k$,

$$\|A_{DI}\| \geq \frac{\|A_{JI}\|}{n\|A_i\|} > \frac{k}{n\|A_i\|} > n \max_{1\leq i\leq m} \|A_i^{-1}\|,$$

namely

$$\frac{n \max\limits_{1\leq i\leq m} \|A_i^{-1}\|}{\|A_{DI}\|} \leq \frac{n^2 \max\limits_{1\leq i\leq m} \|A_i^{-1}\| \max\limits_{1\leq i\leq m} \|A_i\|}{k} < 1.$$

This means that all summands in $f_{\mathcal{A},k}$ and $g_{\mathcal{A},k}$ are $s$-powers of numbers uniformly bounded from above by some number strictly smaller than 1 and therefore $f_{\mathcal{A},k}$ and $g_{\mathcal{A},k}$ are strictly decreasing with $s$ and their image equals $(0, \infty)$. $\square$

**Theorem 1.** *Let $\mathcal{A}$ be a fast gasket of $m$ matrices. Then $0 < s_{\mathcal{A}} < \infty$ and both $s_{g,k} = g^{-1}_{\mathcal{A},k}(1)$ and $s_{f,k} = f^{-1}_{\mathcal{A},k}(1)$ are uniquely defined and converge, respectively from left and right, to $s_{\mathcal{A}}$ as $k \to \infty$ with speed at least logarithmic.*

*Proof.* We start by showing that

$$\zeta_{\mathcal{A}}(s) \geq h_{\mathcal{A},k}(s) + g_{\mathcal{A},k}(s)\zeta_{\mathcal{A}}(s) \tag{16}$$

and

$$\zeta_{\mathcal{A}}(s) \leq h_{\mathcal{A},k}(s) + f_{\mathcal{A},k}(s)\zeta_{\mathcal{A}}(s) \tag{17}$$

for every $k$, where $h_{\mathcal{A},k}$ is some positive continuous function that plays no role here. We prove it in detail for the most complicated case, namely the second one, when we need to pass through the partial sums $\zeta^{\ell}_{\mathcal{A}}$. First we notice that (12) writes as

$$\zeta^{\ell}_{\mathcal{A}_{A_I}}(s) \leq h^0_{\mathcal{A}_{A_I},k} + f^0_{\mathcal{A}_{A_I},k}(s)\zeta^{\ell}_{\mathcal{A}}(s) \tag{18}$$

after putting $h^0_{\mathcal{A}_{A_I},k}(s) = \nu_{\mathcal{A}_{A_I}}(s)$. Now we write the definition of $\zeta^{\ell}_{\mathcal{A}_{A_I}}(s)$ splitting the last term in two

$$\zeta^{\ell}_{\mathcal{A}_{A_I}}(s) = \nu^{\ell}_{\mathcal{A}_{A_I}}(s) + \sum_{\substack{J\in\mathcal{J}^m_{\ell+1}\\ \|A_J\|\leq k}} \zeta^{\ell+1-|J|}_{\mathcal{A}_{A_{JI}}}(s) + \sum_{\substack{J\in\mathcal{J}^m_{\ell+1}\\ \|A_J\|> k}} \zeta^{\ell+1-|J|}_{\mathcal{A}_{A_{JI}}}(s) \tag{19}$$

and apply (12) to the first term in $\zeta$ and (18) to the second, getting

$$\zeta^\ell_{\mathcal{A}_{A_I}}(s) \le \nu_{\mathcal{A}_{A_I}}(s) + \sum_{\substack{J\in\mathcal{J}^m\\ \|A_J\|\le k}} \left[\nu^\ell_{\mathcal{A}_{A_{JI}}}(s) + c_{\mathcal{A}}^{-s}\mu_{\mathcal{A}_{A_{JI}}}(s)\zeta^\ell_{\mathcal{A}}(s)\right] + c_{\mathcal{A}}^{-s} \sum_{\substack{J\in\mathcal{J}^m\\ \|A_J\|>k}} \|A_{JI}\|^{-s}\zeta^\ell_{\mathcal{A}}(s)$$

$$= h^1_{\mathcal{A}_{A_I},k}(s) + c_{\mathcal{A}}^{-s} \sum_{L\in\mathcal{Q}^1_{\mathcal{A},k}} \|A_L\|^{-s}\zeta^\ell_{\mathcal{A}}(s) = h^1_{\mathcal{A}_{A_I},k}(s) + f^1_{\mathcal{A}_{A_I},k}(s)\zeta^\ell_{\mathcal{A}}(s),$$

where we set $h^1_{\mathcal{A}_{A_I},k}(s) = h^0_{\mathcal{A}_{A_I},k}(s) + \sum_{\substack{J\in\mathcal{J}^m\\ \|A_J\|\le k}} \nu_{\mathcal{A}_{A_{JI}}}(s)$.

By repeating recursively this procedure we get that

$$\zeta^\ell_{\mathcal{A}}(s) \le h^r_{\mathcal{A},k}(s) + f^r_{\mathcal{A},k}(s)\zeta_{\mathcal{A}}(s)$$

for all $r \ge 0$. Since the $\mathcal{Q}^r_{\mathcal{A},k}$ stabilize into the $\mathcal{Q}_{\mathcal{A},k}$, we proved that

$$\zeta^\ell_{\mathcal{A}}(s) \le h_{\mathcal{A},k}(s) + f_{\mathcal{A},k}(s)\zeta_{\mathcal{A}}(s).$$

for every $\ell$ and therefore (17) follows; (16) is proved analogously.

By Proposition 7, for $k$ big enough the points $s_{f,k} = f^{-1}_{\mathcal{A},k}(1)$ and $s_{g,k} = g^{-1}_{\mathcal{A},k}(1)$ are uniquely defined and strictly between 0 and $\infty$. An argument analog to the one at the beginning of the section shows that these inequalities imply that $s_{g,k} \le s_{\mathcal{A}} \le s_{f,k}$ for every such $k$. In particular $0 < s_{\mathcal{A}} < \infty$ since every $s_{f,k}$ is finite and every $s_{g,k}$ is larger than 0.

To prove the last part of the theorem we point out that $g_{\mathcal{A},k}(s) = c^s_{\mathcal{A}} n^{-s} f_{\mathcal{A},k}(s)$ and that, for any $s' > 0$,

$$f_{\mathcal{A},k}(s) = \frac{m-1}{c^s_{\mathcal{A}}} \sum_{J\in\mathcal{Q}_{\mathcal{A},k}} \|A_J\|^{-s} > (c_{\mathcal{A}}k)^{s'} \frac{m-1}{c_{\mathcal{A}}^{-s-s'}} \sum_{J\in\mathcal{Q}_{\mathcal{A},k}} \|A_J\|^{-s-s'} = (c_{\mathcal{A}}k)^{s'} f_{\mathcal{A},k}(s').$$

Hence

$$\begin{aligned} 1 = g_{\mathcal{A},k}(s_{g,k}) = c^{s_{g,k}} n^{-s_{g,k}} f_{\mathcal{A},k}(s_{g,k}) >\\ > c_{\mathcal{A}}^{s_{g,k}} n^{-s_{g,k}} (c_{\mathcal{A}}k)^{s_{f,k}-s_{g,k}} f_{\mathcal{A},k}(s_{f,k}) = c_{\mathcal{A}}^{s_{f,k}} n^{-s_{g,k}} k^{s_{f,k}-s_{g,k}} \end{aligned}$$

so that

$$0 > s_{f,k}(\log k + \log c_{\mathcal{A}}) - s_{g,k}(\log k + \log n)$$

and finally

$$0 \le s_{f,k} - s_{g,k} < s_{g,k}\left(\frac{\log k + \log n}{\log k + \log c_{\mathcal{A}}} - 1\right) \le s_{\mathcal{A}} \frac{\log n - \log c_{\mathcal{A}}}{\log k + \log c_{\mathcal{A}}}.$$

□

### 3.2.3 An alternate characterization of $s_{\mathcal{A}}$

The exponent $s_{\mathcal{A}}$ of a $m$-gasket $\mathcal{A}$ can also be extracted from the asymptotics of the partial sums of the $\|A_I\|^{-s}$ over same-rank multi-indices, namely from the sequence of functions $\zeta_{\mathcal{A},k}(s) = \sum_{I\in\mathcal{I}^m_k} \|A_I\|^{-s}$.

**Lemma 1.** *The sequence of analytical log-convex monotonically decreasing functions $\zeta^{1/k}_{\mathcal{A},k}(s)$ converges pointwise for every $s \in [0,\infty)$ to a bounded continuous log-convex monotonically decreasing function $\xi_{\mathcal{A}}(s)$.*

*Proof.* The $\zeta^{1/k}_{\mathcal{A},k}(s)$ are analytical log-convex monotonically decreasing functions because every summand $\|A_I\|^{-s}$ of the $\zeta_{\mathcal{A},k}$ satisfies those properties and so does every finite or infinite (converging) sum of them and positive power.

In order to prove the convergence of the sequence we can replace without loss of generality in the expression of the $\zeta_{\mathcal{A},k}$ (by abuse of notation we will denote the new functions still by $\zeta_{\mathcal{A},k}$) the max norm with any submultiplicative norm $\|\cdot\|'$ and notice that, since $\|A_{IJ}\|' \leq \|A_I\|'\|A_J\|'$, then $\zeta_{\mathcal{A},k+k'}(s) \geq \zeta_{\mathcal{A},k}(s)\zeta_{\mathcal{A},k'}(s)$. From this follows that, for every $s$, the sequence $\zeta^{1/k}_{\mathcal{A},k}(s)$ can have only one accumulation point and this point must be equal to $\sup_{k\in\mathbb{N}} \zeta^{1/k}_{\mathcal{A},k}(s)$. The main point is that for every element $\zeta^{1/k_0}_{\mathcal{A},k_0}(s)$ almost all other elements of the sequence are not smaller than $\zeta^{1/k}_{\mathcal{A},k_0}(s) - \varepsilon$ for every $\varepsilon > 0$. Indeed if $k = Nk_0$ then immediately $\zeta^{1/k}_{\mathcal{A},k}(s) \geq \left[\zeta^{N}_{\mathcal{A},k_0}(s)\right]^{1/k_0} = \zeta^{1/k_0}_{\mathcal{A},k_0}(s)$, while if $k = Nk_0 + \ell$, with $1 \leq \ell \leq k_0 - 1$, then $\zeta_{\mathcal{A},k}(s) \geq \zeta^{N}_{\mathcal{A},k_0}(s)\zeta_{\mathcal{A},\ell}(s)$, so that

$$\zeta^{1/k}_{\mathcal{A},k}(s) \geq \zeta^{\frac{N}{Nk_0+\ell}}_{\mathcal{A},k_0}(s)\zeta^{\frac{1}{Nk_0+\ell}}_{\mathcal{A},\ell}(s) = \left[\zeta^{1/k_0}_{\mathcal{A},k_0}(s)\right]^{\frac{1}{1+\ell/(Nk_0)}} \zeta^{\frac{1}{Nk_0+\ell}}_{\mathcal{A},\ell}(s)$$

Since there are only a finite number of possible values of $\ell$, for every $\varepsilon' > 0$ we can find a $N$ big enough s.t. both $\left|\left[\zeta^{1/k_0}_{\mathcal{A},k_0}(s)\right]^{\frac{1}{1+\ell/(Nk_0)}} - \zeta^{1/k_0}_{\mathcal{A},k_0}(s)\right| < \varepsilon'$ and $\left|\zeta^{\frac{1}{Nk_0+\ell}}_{\mathcal{A},\ell}(s) - 1\right| < \varepsilon'$ hold for all $\ell$. Hence

$$\zeta^{1/k}_{\mathcal{A},k}(s) \geq \zeta^{1/k_0}_{\mathcal{A},k_0}(s) - \varepsilon'\left(\zeta^{1/k_0}_{\mathcal{A},k_0}(s) + 1 - \varepsilon'\right) \geq \zeta^{1/k_0}_{\mathcal{A},k_0}(s) - \varepsilon$$

for small enough $\varepsilon'$.

That $\xi_{\mathcal{A}}(s) = \sup_{k\in\mathbb{N}} \zeta^{1/k}_{\mathcal{A},k}(s)$ be finite for all $s$ is clear from the fact that all $\zeta^{1/k}_{\mathcal{A},k}(s)$ are positive decreasing functions bounded by $\zeta^{1/k}_{\mathcal{A},k}(0) = m$. $\square$

**Theorem 2.** *The function $\xi_{\mathcal{A}}$ satisfies the following properties:*

1. $\xi_{\mathcal{A}}(s) > 1$ *for* $s < s_{\mathcal{A}}$*;*
2. $\xi_{\mathcal{A}}(s_{\mathcal{A}}) = 1$*;*
3. $\xi_{\mathcal{A}}(s) < 1$ *for* $s > s_{\mathcal{A}}$ *if* $\mathcal{A}$ *is a hyperbolic gasket;*
4. $\xi_{\mathcal{A}}(s) = 1$ *for* $s > s_{\mathcal{A}}$ *if* $\mathcal{A}$ *is a parabolic fast gasket.*

*Proof.* Directly from the $n$-th root test we get that $\xi_{\mathcal{A}}(s) \geq 1$ for $s < s_{\mathcal{A}}$ and $\xi_{\mathcal{A}}(s) \leq 1$ for $s > s_{\mathcal{A}}$, so that in particular $\xi_{\mathcal{A}}(s_{\mathcal{A}}) = 1$.

Assume first that $\mathcal{A}$ is hyperbolic, so that there exist constants $\alpha > 1$ and $K > 0$ s.t. $\|A_I\| \geq K\alpha^{|I|}$ for every $I \in \mathcal{I}^m$. Hence

$$\frac{d}{ds}\ln\zeta_k^{1/k}(s) = -\frac{1}{k}\frac{\sum_{|I|=k}\left(\|A_I\|^{-s}\ln\|A_I\|\right)}{\sum_{|I|=k}\|A_I\|^{-s}} \leq -\ln\alpha - \frac{\ln K}{k},$$

namely for every $\varepsilon > 0$ we can find a $\alpha' > 1$, with $|\alpha - \alpha'| \leq \varepsilon$, and a $\bar{k} > 0$ such that $(\ln\zeta_k^{1/k}(s))' \leq -\ln\alpha'$ for all $k \geq \bar{k}$. Since $\log_\alpha \zeta_k^{1/k}(s_{\mathcal{A}}) = 0$ this means that, for every $k \in \mathbb{N}$ and $s > 0$, $\ln\zeta_k^{1/k}(s_{\mathcal{A}} + s) \leq -s\ln\alpha'$ and $\ln\zeta_k^{1/k}(s_{\mathcal{A}} - s) \geq s\ln\alpha'$, namely $\zeta_k^{1/k}(s_{\mathcal{A}}) \geq (\alpha')^s > 1$ at the left of $s_{\mathcal{A}}$ and $\zeta_k^{1/k}(s_{\mathcal{A}}) \leq (\alpha')^{-s} < 1$ at its right. Since this is true for almost all $k$, the same properties hold for $\xi_{\mathcal{A}}$.

Assume now that $\mathcal{A}$ is fast parabolic and that $s_{\mathcal{A}} < \infty$ (e.g. in case that $\mathcal{A}$ is fast). In this case the sequence $a_k = \min_{|I|=k}\{\|A_I\|\}$ grows polynomially and therefore, for $s > s_{\mathcal{A}}$,

$$1 \geq \zeta_k^{1/k}(s) \geq a_k^{-s/k} = \left(a_k^{-\frac{1}{k}}\right)^{-s} \overset{k\to\infty}{\longrightarrow} 1,\ \text{i.e.}\ \xi_{\mathcal{A}}(s) = \lim_{k\to\infty}\zeta_k^{1/k}(s) = 1.$$

Proving that $\xi_{\mathcal{A}}(s) > 1$ for $s < s_{\mathcal{A}}$ requires much more work. We start by observing that, analogously to (3) and (2), we have the inequality

$$\zeta_{\mathcal{A}_{A_I},k}(s) \geq \frac{1}{n^s\|A_I\|^s}\zeta_{\mathcal{A},k}(s) \tag{20}$$

and we can re-write $\zeta_{\mathcal{A}_{A_I},k}$ as follows:

$$\zeta_{\mathcal{A}_{A_I},k}(s) = \sum_{\substack{D\in\mathcal{D}_k^m\\|D|=k}}\zeta_{\mathcal{A}_{A_{DI}},0}(s) + \sum_{J\in\mathcal{J}_k^m}\zeta_{\mathcal{A}_{A_{JI}},k-|J|}(s). \tag{21}$$

Applying (20) to (21) we get that

$$\zeta_{\mathcal{A}_{A_I},k}(s) \geq \sum_{\substack{D\in\mathcal{D}_k^m\\|D|=k}} \zeta_{\mathcal{A}_{A_{DI}},0}(s) + n^{-s} \sum_{J\in\mathcal{J}_k^m} \|A_{JI}\|^{-s}\zeta_{\mathcal{A},k-|J|}(s).$$

We set $w^{0,j}_{\mathcal{A}_{A_I},\kappa}(s) = \sum_{\substack{J\in\mathcal{J}_k^m\\|J|=j}} n^{-s}\|A_{JI}\|^{-s}$, so that

$$\zeta_{\mathcal{A}_{A_I},k}(s) \geq \sum_{2\leq j\leq k} w^{0,j}_{\mathcal{A}_{A_I},\kappa}(s)\zeta_{\mathcal{A},k-j}(s). \tag{22}$$

Now we split the second summation in (21) as we did in (19) and obtain

$$\zeta_{\mathcal{A}_{A_I},k}(s) = \sum_{\substack{D\in\mathcal{D}_k^m\\|D|=k}} \zeta_{\mathcal{A}_{A_{DI}},0}(s) + \sum_{\substack{J\in\mathcal{J}_k^m\\\|A_J\|\leq\kappa}} \zeta_{\mathcal{A}_{A_{JI}},k-|J|}(s) + \sum_{\substack{J\in\mathcal{J}_k^m\\\|A_J\|>\kappa}} \zeta_{\mathcal{A}_{A_{JI}},k-|J|}(s).$$

Now we apply recursively (22) to the first term in the splitting and (20) to the second, obtaining

$$\zeta_{\mathcal{A}_{A_I},k}(s) \geq \sum_{\substack{J\in\mathcal{J}_k^m\\\|A_J\|\leq\kappa}} \zeta_{\mathcal{A}_{A_{JI}},k-|J|}(s) + \sum_{\substack{J\in\mathcal{J}_k^m\\\|A_J\|>\kappa}} \zeta_{\mathcal{A}_{A_{JI}},k-|J|}(s) \geq$$

$$\geq \sum_{2\leq j\leq k} \sum_{\substack{J\in\mathcal{J}_k^m\\\|A_J\|\leq\kappa}} w^{0,j}_{\mathcal{A}_{A_{JI}},\kappa}(s)\zeta_{\mathcal{A},k-j}(s) + n^{-s} \sum_{\substack{J\in\mathcal{J}_k^m\\\|A_J\|>\kappa}} \|A_{JI}\|^{-s}\zeta_{\mathcal{A},k-|J|}(s) =$$

$$= \sum_{2\leq j\leq k} w^{1,j}_{\mathcal{A}_{A_I},\kappa}(s)\zeta_{\mathcal{A},k-j}(s),$$

where

$$w^{1,j}_{\mathcal{A}_{A_I},\kappa}(s) = \sum_{\substack{J\in\mathcal{J}_k^m\\|J|=j\\\|A_J\|\leq\kappa}} w^{0,j}_{\mathcal{A}_{A_{JI}},\kappa}(s) + n^{-s} \sum_{\substack{J\in\mathcal{J}_k^m\\|J|=j\\\|A_J\|>\kappa}} \|A_{JI}\|^{-s}.$$

Since there are only finitely many terms in $\mathcal{A}$ with norm not larger than $\kappa$, by repeating recursively this procedure, the $w^{i,j}_{\mathcal{A}_{A_I},\kappa}$ stabilize to some functions $w^{j}_{\mathcal{A}_{A_I},\kappa}$ such that $\zeta_{\mathcal{A}_{A_I},k}(s) \geq \sum_{2\leq j\leq k} w^{j}_{\mathcal{A}_{A_I},\kappa}(s)\zeta_{\mathcal{A},k-j}(s)$. For $I = 0$ we get

$$\zeta_{\mathcal{A},k}(s) \geq \sum_{2\leq j\leq k} w^{j}_{\mathcal{A},\kappa}(s)\zeta_{\mathcal{A},k-j}(s).$$

Now consider the polynomials $p_k(x) = x^k - \sum_{2\le j\le k} w^j_{\mathcal{A},\kappa}(s)x^{k-j}$. By Descartes' rule of signs they all have a single positive root. Moreover for $k$ big enough this root is larger than 1. Indeed by comparing the recursive algorithm that generate the $w^j_{\mathcal{A},\kappa}$ with the one generating $g_{\mathcal{A},\kappa}$ it is clear that the function $W^k_{\mathcal{A}_{A_I},\kappa}(s) = \sum_{2\le j\le k} w^j_{\mathcal{A}_{A_I},\kappa}(s)$ is equal to the truncation of the series $g_{\mathcal{A}_{A_I},\kappa}$ to the matrices $A_I$ with $|I| \le k$. In particular this means that when $s < s_{\mathcal{A}}$ there is some $\bar{k}$ large enough that $W^{\bar{k}}_{\mathcal{A}_{A_I},\kappa}(s) > 1$. Then $p_{\bar{k}}(1) < 0$ and therefore the positive root $\sigma$ of $p_{\bar{k}}$ is larger than 1. Hence

$$\mu \stackrel{\text{def}}{=} \min_{0\le j\le \bar{k}} \{\zeta_{\mathcal{A},j}(s)\sigma^{-j}\} = \inf_{0\le j\le \infty} \{\zeta_{\mathcal{A},j}(s)\sigma^{-j}\},$$

which follows by induction as a consequence of the following observation:

$$\zeta_{\mathcal{A},\bar{k}+1}(s) \ge \sum_{2\le j\le \bar{k}} w^j_{\mathcal{A}_{A_I},\kappa}(s)\zeta_{\mathcal{A},\bar{k}+1-j}(s) \ge \mu \sum_{2\le j\le \bar{k}} w^j_{\mathcal{A}_{A_I},\kappa}(s)\sigma^{\bar{k}+1-j} \ge \mu\sigma^{\bar{k}+1}.$$

Hence $\lim_{j\to\infty} \zeta^{1/j}_{\mathcal{A},j}(s) \ge \sigma > 1$ for every $s \ge s_{\mathcal{A}}$. □

**Example 9.** *Let $M_1, \dots, M_m$ be upper triangular matrices of the form*

$$M_i = \begin{pmatrix} \alpha_i & \beta_i \\ 0 & 1 \end{pmatrix}$$

*and assume that*

$$\max_{1\le i\le m} \{\beta_i\} \le \frac{1}{1 - \max_{1\le i\le m}\{1-\alpha_i\}}.$$

*It is easy to prove by induction that under this assumption the non-zero off-diagonal term never gets larger than 1, so that $\|M_I\| = 1$ for every $I \in \mathcal{I}^m$.*

*Now consider the gasket $\mathcal{A}$ generated by $A_i = \rho_i M_i$, $\rho_i > 1$. By the observation above for every $I = i_1 \dots i_k$ we have that $\|A_I\| = \rho_{i_1}\cdots\rho_{i_k}$ and therefore*

$$\zeta_{\mathcal{A},k} = \sum_{I\in\mathcal{I}^m_k} \|A_I\|^{-s} = \sum_{I\in\mathcal{I}^m_k} \rho^{-s}_{i_1}\cdots\rho^{-s}_{i_k} = \left(\rho_1^{-s} + \cdots + \rho_m^{-s}\right)^k.$$

*Since $\mathcal{A}$ is clearly a hyperbolic gasket, by Theorem 2 its exponent $s_{\mathcal{A}}$ is the unique solution of the equation $\rho_1^{-s} + \cdots + \rho_m^{-s} = 1$. Similar but more complicated conditions can be found for upper triangular matrices in higher dimension (see Section 4.3.1 for a similar case with $3\times 3$ matrices).*

### 3.2.4 Norm asymptotics of fast gaskets

We can use the previous section's results to prove Theorem 2.

**Lemma 2.** *Let $\mathcal{A} : \mathcal{I}^m \to M_n(K)$ be a semigroup homomorphism and $M \in GL_n(K)$. Then*

$$N_{\mathcal{A}_M}(r) > N_{\mathcal{A}}\left(\frac{r}{n\|M\|}\right). \tag{23}$$

*Proof.* Since $\|AM\| \leq n\|A\|\|M\|$, we have that $\|A_I\| \leq \frac{r}{n\|M\|} \implies \|A_I M\| \leq r$, namely $\{A_I | \|A_I\| \leq \frac{r}{n\|M\|}\} \subset \{A_I M | \|A_I M\| \leq r\}$ □

**Theorem 3.** *Let $\mathcal{A}$ be a hyperbolic or fast parabolic gasket. Then*

$$\lim_{r\to\infty} \frac{\log N_{\mathcal{A}_M}(r)}{\log r} = s_{\mathcal{A}}.$$

*for every $M \in GL_n(K)$.*

*Proof.* Since $\|A\|/(n\|M^{-1}\|) \leq \|AM\| \leq n\|A\|\|M\|$ we can prove the theorem without loss of generality for $M = \mathbb{1}_n$.

$\boldsymbol{\limsup_{r\to\infty} \frac{\log N_{\mathcal{A}}(r)}{\log r} \leq s_{\mathcal{A}}}$. Let $s > s_{\mathcal{A}}$. Then

$$\infty > \zeta_{\mathcal{A}}(s) > \sum_{\|A_I\|\leq r} \|A_I\|^{-s} \geq \sum_{\|A_I\|\leq r} r^{-s} = N_{\mathcal{A}}(r) r^{-s},$$

so that

$$s + \frac{\log \zeta_{\mathcal{A}}(s)}{\log r} > \frac{\log N_{\mathcal{A}}(r)}{\log r}$$

and therefore $\limsup_{r\to\infty} \frac{\log N_{\mathcal{A}}(r)}{\log r} \leq s$. Since this is true for every $s > s_{\mathcal{A}}$ it follows at once that $\limsup_{r\to\infty} \frac{\log N_{\mathcal{A}}(r)}{\log r} \leq s_{\mathcal{A}}$.

$\boldsymbol{\liminf_{r\to\infty} \frac{\log N_{\mathcal{A}}(r)}{\log r} \geq s_{\mathcal{A}}}$. From the elementary observation that

$$\{A_I | \|A_I\| \leq r, I \in \mathcal{I}^m\} \supset \bigcup_{J\in\mathcal{I}^m_k} \{A_{IJ} | \|A_{IJ}\| \leq r, I \in \mathcal{I}^m\}$$

and using (23) we get that, for every $k \in \mathbb{N}$,

$$N_{\mathcal{A}}(r) \geq \sum_{J\in\mathcal{I}^m_k} N_{\mathcal{A}_{A_J}}(r) \geq \sum_{J\in\mathcal{I}^m_k} N_{\mathcal{A}}\left(\frac{r}{n\|A_J\|}\right).$$

Assume now $s > s_{\mathcal{A}}$. Since $\mathcal{A}$ is, by hypothesis, either a hyperbolic or a fast parabolic gasket, by the definition of gasket and Theorem 2 we can always choose a $k_0$ such that $\|A_I\| > 1/n$ for $|I| \geq k_0$ and $\sum_{I\in\mathcal{I}^m_{k_0}} \|A_I\|^{-s} > n^s$.

Now set $a_m = n \min_{I\in\mathcal{I}^m_{k_0}} \|A_I\|$ and $a_M = n \max_{I\in\mathcal{I}^m_{k_0}} \|A_I\|$, let $r_0 > 0$ be s.t. $N_{\mathcal{A}}(r_0) > 0$ and set $r_1 = a_M r_0$ and $r_i = a_m^{i-1} r_1$, $i \geq 2$. Similarly to the proof of Theorem 2 we have that, by induction,

$$M \stackrel{\text{def}}{=} \min_{r\in[r_0,r_1]} N_{\mathcal{A}}(r_0) r^{-s} = \inf_{r\in[r_0,\infty]} N_{\mathcal{A}}(r_0) r^{-s}.$$

Indeed note first of all that $\lim_{i\to\infty} r_i = \infty$ since we chose $k_0$ so that $a_m > 1$. Assume now that $N_{\mathcal{A}}(r) \geq M r^s$ in $[r_0, r_i]$ and let $r \in [r_i, r_{i+1}]$. Then for every $J \in \mathcal{I}^m_{k_0}$ we have that $r/(n\|A_J\|) \in [r_0, r_i]$ since

$$r_i = \frac{r_{i+1}}{a_m} \geq \frac{r}{n\|A_J\|} \geq \frac{r_i}{a_M} = a_m^{i-1} r_0 \geq r_0.$$

and therefore

$$N_{\mathcal{A}}(r) \geq \sum_{J\in\mathcal{I}^m_k} N_{\mathcal{A}}\left(\frac{r}{n\|A_J\|}\right) \geq \sum_{J\in\mathcal{I}^m_k} M \left[\frac{r}{n\|A_J\|}\right]^s \geq M r^s.$$

Hence it follows at once that

$$\frac{\log N_{\mathcal{A}}(r)}{\log r} \geq \frac{\log M}{\log r} + s$$

and therefore $\liminf_{r\to\infty} \frac{logN_{\mathcal{A}}(r)}{\log r} \geq s$. Since this is true for all $s < s_{\mathcal{A}}$ it follows that $\liminf_{r\to\infty} \frac{logN_{\mathcal{A}}(r)}{\log r} \geq s_{\mathcal{A}}$. □

# 4 Hausdorff dimension of limit sets of discrete subsemigroups of real and complex projective automorphisms.

In this section we show how the exponent of a free finitely generated semigroup $\boldsymbol{A} \subset SL_n(K)$ (resp. $\boldsymbol{A} \subset SL^{\pm}_n(K)$ if $n = 2n'$ and $K = \mathbb{R}$) is sometimes related to the Hausdorff dimension of the set of limit points of a generic orbit in $KP^{n-1}$ of the subsemigroup $\Psi(\boldsymbol{A}) \subset PSL_n(K)$ (resp.

$\Psi(\boldsymbol{A}) \subset PSL^{\pm}_{2n'}(\mathbb{R})$) naturally induced by $\boldsymbol{A}$ (equivalently, to the residual set of the IFS corresponding to $\Psi(\boldsymbol{A})$).

Throughout this section we will make analytical and numerical evaluations of the exponent of several semigroups. The analytical bounds are obtained via the functions $\mu_{\boldsymbol{A},k}$ defined in Section 3.2.2 thanks to Theorem 1. The numerical ones are obtained by evaluating numerically the function $N_{\boldsymbol{A}}(k)$ and interpolating the curve $\log N_{\boldsymbol{A}}(k)$ with respect to $\log k$ thanks to Theorem 3. Since $\boldsymbol{A}$ is a gasket and $N_{\boldsymbol{A}}(k)$ is integer-valued, a computer program can evaluate *exactly* its values, the only constraint coming from the running time which increases exponentially with $k$. All calculations were done with Xeon 2MHz CPUs under Linux.

## 4.1 $n = 2, K = \mathbb{R}$

Let $\{f_1, \dots, f_m\}$ be a free set of linear automorphisms of $\mathbb{R}^2$ preserving a volume 2-form *modulo sign*. With respect to any frame $\mathcal{E} = \{e_1, e_2\}$, these automorphisms are represented by matrices $A_i \in SL^{\pm}_2(\mathbb{R})$. We denote by $\boldsymbol{A}$ the semigroup generated by the $A_i$ and by $\psi_I \in PSL^{\pm}_2(\mathbb{R})$ the automorphism of $\mathbb{R}P^1 \simeq \mathbb{S}^1$ naturally induced by $f_I$, $I \in \mathcal{I}^2$. The similarity between the characterization of the exponent $s_{\mathcal{A}}$ given in Theorem 2 and the formula for the Hausdorff dimension of a 1-dimensional IFS given in [Fal90] (Theorem 9.9, p. 126) suggests the following claim:

**Theorem 4.** *Assume that the $f_i$ have all real distinct eigenvalues and that there exists some proper open set $V \subset \mathbb{R}P^1$ invariant by the $\psi_i$ and such that, for some affine chart $\varphi : \mathbb{R}P^1 \to \mathbb{R}$, the $\psi_i$ are contractions on $\varphi(\overline{V})$ with respect to the Euclidean distance and satisfy the "open set condition" $\psi_1(V) \cap \psi_2(V) = \emptyset$. Let $R_{\boldsymbol{A}} = \cap_{k=1}^{\infty} \left(\cup_{|I|=k} \psi_I(V)\right)$ be the corresponding residual set. Then $2 \dim_H R_{\boldsymbol{A}} = s_{\boldsymbol{A}}$.*

*Proof.* Let $(x, y)$ and $[x : y]$ the affine and homogeneous coordinates associated to $\varphi$ and assume, for the argument's sake, that $\varphi([x : y]) = x/y$. Under the hypotheses every $\psi_I$ has two fixed points, an attractive one $a_I$ and a repulsive one $r_I$. Since we are assuming $V$ to be invariant under the $\psi_I$, then it must happen that $a_I \in \overline{V}$ for all $I \in \mathcal{I}^m$. Let $A_I = \begin{pmatrix} \alpha & \beta \\ \mu & \nu \end{pmatrix}$ be the matrix representing the $f_I$ in the affine coordinates relative to $\varphi$. Let $\Psi_I = \varphi \circ \psi_I \circ \varphi^{-1}(x)$ be the coordinate expression of $\psi_I$ in the chart $\varphi$. Then

a direct calculation shows that

$$\Psi_I(\varphi) = \frac{\alpha\varphi+\beta}{\mu\varphi+\nu},\ \Psi'_I(\varphi) = \frac{\det A_I}{(\mu\varphi+\nu)^2},\ \left|\Psi'_I(\varphi(a_I))\right| = \frac{1}{\|A_I\|^2}$$

(recall that $\det A_I = \pm 1$). Now let $\varphi(\overline{V}) = [\varphi_1, \varphi_2]$ and set $\varphi_m = (\varphi_1+\varphi_2)/2$. Then $\|A_I\|/2 \leq \mu\varphi_m + \nu \leq \|A_I\|$ and therefore

$$\frac{1}{\|A_I\|^2} \leq \left|\Psi'_I(\varphi_m)\right| \leq \frac{4}{\|A_I\|^2}.$$

Hence the limit

$$\lim_{k\to\infty} \left[\sum_{|I|=k} \left|\Psi'_I(\varphi_m)\right|^s\right]^{\frac{1}{k}}$$

converges iff it converges the limit

$$\lim_{k\to\infty} \left[\sum_{|I|=k} \|A_I\|^{-2s}\right]^{\frac{1}{k}}.$$

By the result on 1-dimensional contractions quoted above, the exponent separating the values of $s$ for which the first limit diverge from those for which it converges is exactly $\dim_H R_{\boldsymbol{A}}$. By Theorem 2 this means exactly that $2\dim_H R_{\boldsymbol{A}} = s_{\boldsymbol{A}}$. □

An interesting consequence of the previous theorem is the following constraint posed by geometry to the (algebraic) exponent of the semigroups satisfying its conditions:

**Corollary 2.** *Let $\boldsymbol{A} \subset SL^{\pm}_2(\mathbb{R})$ be a semigroup satisfying the conditions of the theorem above. Then $s_{\boldsymbol{A}} \leq 2$.*

*Proof.* This is a direct consequence of the trivial fact that the Hausdorff dimension of a subset of $\mathbb{R}$ cannot be bigger than 1. □

### 4.1.1 Matrices with non-negative entries

The semigroup $SL^{\pm}_2(\mathbb{R}^+)$ is a source of several interesting semigroups that satisfy the hypothesis of Theorem 4. Indeed every linear automorphism

$f = A^i_j e_i \otimes \varepsilon^j$ preserves the positive cone $C(\mathcal{E})$ over $\mathcal{E}$ and therefore the induced projective automorphism $\psi_f$ preserves the closed segment $S(\mathcal{E}) \subset \mathbb{R}P^1$.

In this simple setting there is an easy sufficient condition to determine whether a gasket is fast:

**Proposition 8.** *Let $\mathcal{A} : \mathcal{I}^m \to SL^{\pm}_2(\mathbb{R}^+)$ be a gasket and suppose that all products $A_I$, $|I| = 2$, have no entry equal to zero. Then $\mathcal{A}$ is fast.*

*Proof.* Let $A_{12} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and $A_K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that

$$A_{12K} = \begin{pmatrix} pa + qc & pb + qd \\ ra + sc & rb + bd \end{pmatrix}.$$

Assume, for the argument's sake, that $\|A_{12K}\| = ra + sb$. Then, since

$$pa + qc \geq \frac{\min\{p, q\}}{\max\{r, s\}}(ra + sb),$$

we have that, for every $M \in M_2(\mathbb{R}^+)$,

$$\|MA_{12K}\| \geq \|M\|(pa + qc) \geq \frac{\min\{p, q\}}{\max\{r, s\}}\|M\|\|A_{12K}\|.$$

By repeating this argument for every index of order 2 and denoting by $c$ the smallest of these coefficients, we have that $\|MA_{12K}\| \geq c\|M\|\|A_{12K}\|$ for every $M \in M_2(\mathbb{R}^+)$. In particular then $\mathcal{A}$ is a fast gasket with coefficient not smaller than $c$. □

**Example 10.** *Let $\mathcal{E} = \{e_1, e_2\}$ be a frame on $\mathbb{R}^2$ and $f_{1,2}$ defined by*

$$f_1(e_1) = e_1 + e_2, f_1(e_2) = e_2;\; f_2(e_1) = 2e_1 + e_2, f_2(e_2) = e_2.$$

*With respect to $\mathcal{E}$ the $f_i$ are represented by the matrices*

$$F_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, F_2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

*The semigroup $\boldsymbol{F} = \langle F_1, F_2 \rangle \subset SL^-(\mathbb{N})$ is free because if $F_I \in \boldsymbol{A}$, $I \in \mathcal{I}^2$, then the entries in $F_I$'s lower row are equal to the entries in the upper row of the matrix $F_{I'}$ and according to whether the upper left entry of $F_I$ is larger*

*or smaller than its lower left entry we get whether $I = 2I'$ or $I = 1I'$. Proceeding recursively this way we see that there is no other index $J \neq I$ s.t. $F_J = F_I$. In particular then $\boldsymbol{F}$ is a gasket. Moreover $\boldsymbol{F}$ is hyperbolic: indeed clearly $\|F_I\| \geq \|F_1^{|I|}\|$, since $F_2$ has no entry smaller than the corresponding entry of $F_1$, and $\|F_1^k\| \simeq g^k$, where $g$ is the golden ratio, because clearly $F_1^k = \begin{pmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{pmatrix}$, where $f_k$, $k \geq 1$, is the Fibonacci sequence $0, 1, 1, 2, 3, 5, \dots$ Finally, $\boldsymbol{F}$ is fast (with coefficient not smaller than $1/3$) by the previous proposition.*

*A direct calculation shows that $\psi_{1,2}$ are not contractions over $S(\mathcal{E})$ but they are so over the smaller set $S(\mathcal{E}')$, with*

$$\mathcal{E}' = \{e_1' = (1+\sqrt{3}, 2), e_2' = (1+\sqrt{3}, 1)\}.$$

*Let $[x : y]$ be homogeneous coordinates on $\mathbb{R}P^1$ corresponding to $\mathcal{E}'$. In the canonical chart $\varphi = x/y$ the maps $\psi_i$ induced by $f_i$ write as*

$$\psi_1(\varphi) = \frac{\varphi+1}{\varphi},\ \psi_2(\varphi) = \frac{2\varphi+1}{\varphi},$$

*which reveals that this example coincides with Example 9.8 of [Fal90], coming from the theory of continued fractions.*

*To obtain analytical bounds for $s_{\boldsymbol{F}}$ we can use Theorem 2. Since both generators have an eigenvalue larger than 1 the norms of the terms $F_1F_2^k$ and $F_2F_1^k$ grow exponentially, so that we can get a good approximation of $\mu_{\boldsymbol{F},\ell}$ by truncating the sums after just a few terms. By considering only the terms with $k \leq 10$ in $\mu_{\boldsymbol{F},0}$ and solving the equation $\mu_{\boldsymbol{F},0}(s) = 2^s$ in this approximation we get $s_{\boldsymbol{F}} \geq .51$, with a relative error of about 6% on the more precise estimate $s_{\boldsymbol{F}} \geq .54$ obtained by considering $k \geq 20$. Since $c = 1/3$, the first $\mu_{\boldsymbol{F},\ell}$ we can get upper bounds is $\mu_{\boldsymbol{F},3}$. Here we just mention that from $\mu_{\boldsymbol{F},8}$, considering the first 30 summands of all series that appear in its expression, we get $0.95 \leq s_{\boldsymbol{F}} \leq 1.76$. In terms of the dimension of $R_{\boldsymbol{F}}$ this translates in $0.474 \leq \dim_H R_{\boldsymbol{F}} \leq 0.877$. By evaluating $N_{\boldsymbol{F}}(k)$ for $k = 2^p$, $1 \leq p \leq 28$, (taking about 20 minutes of CPU time) we get the estimate $s_{\boldsymbol{F}} \simeq 1.062$ (see Table 3 for the corresponding values of $N_{\boldsymbol{F}}$), with a (heuristic) error of 2 on the last digit. This corresponds to the well-known fact $\dim_H R_{\boldsymbol{F}} \simeq 0.531$.*

**Example 11.** *Consider now*

$$f_1(e_1) = e_1, f_1(e_2) = e_1 + e_2;\ f_2(e_1) = e_1 + e_2, f_2(e_2) = e_2.$$

*The corresponding matrices (with respect to $\mathcal{E}$)*

$$C_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

*generate the semigroup $\boldsymbol{C}_2 \subset SL_2(\mathbb{N})$ we already met in Examples 3 and 7. In particular we already know that $\boldsymbol{C}_2$ is a parabolic fast gasket with coefficient $c = 1/2$. A direct check shows that the slowest and fastest growths with respect to the order $k$ of the multi-index $I$ of $C_I \in \mathcal{A}$ correspond respectively to the pure powers $C_i^k$, for which $\|C_i^k\| = k$, and to the "cyclic" products $C_iC_{i+1}C_{i+2}\cdots C_{i+k-1}$, for which $\|C_iC_{i+1}C_{i+2}\cdots C_{i+k-1}\| \simeq g^k$, where the sums in the indices are meant "modulo 2" in the sense that 3 means 1, 4 means 2 and so on. The reason why the golden ration $g$ appears is that, similarly to the previous case,*

$$C_1C_2C_1\cdots C_{i+k-1} = \begin{pmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{pmatrix}$$

*for $k$ odd while if $k$ is even the two rows get exchanged and analogously for the cyclic products beginning by $C_2$.*

*In the affine chart $\varphi : [x : y] \to x/(x + y)$ the maps $\psi_i$ induced by the $f_i$ write as*

$$\psi_1(\varphi) = \frac{\varphi}{1+\varphi},\ \psi_2(\varphi) = \frac{1}{2-\varphi}$$

*and the segment $S(\mathcal{E})$ maps into $[0, 1]$. Note that this choice of chart corresponds to writing $e_1 = e_1'$ and $e_2 = e_1' + e_2'$, expressing the $f_i$ with respect to $\mathcal{E}' = \{e_1', e_2'\}$ and using the canonical chart $y' = 1$ for the corresponding homogeneous coordinates $[x' : y']$. In terms of the semigroup, this corresponds to the adjunction via the matrix $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. A direct calculation shows that the $|\psi_i'(\varphi)| \leq 1$ on $[0, 1]$, with the equal sign holding at 0 for $\psi_1$ and at 1 for $\psi_2$, namely the IFS $\{\psi_1, \psi_2\}$ is parabolic.*

*Evaluating the Hausdorff dimension of the limit set $R_{\mathcal{C}_2}$ of a point $w \in (0, 1)$ under the action of $\mathcal{C}_2$ is nevertheless an easy task. Indeed, since $\psi_1((0, 1)) = (0, 1/2)$ and $\psi_2((0, 1)) = (1/2, 1)$, the $\psi_I$, $|I| = k$, subdivide $(0, 1)$ into $2^k$ disjoint segments $d_I = \psi_I(0, 1)$ in such a way that $\cup_{|I|=k}\overline{d_I} = [0, 1]$. Moreover the length of these segments goes to zero for $k \to \infty$. Indeed if*

$C_I = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{C}_2$ *then* $\psi_I(\varphi) = \frac{(a-c)\varphi+c}{(a+b-c-d)\varphi+c+d}$ *and therefore*

$$|d_I| = \left|\frac{c}{c+d} - \frac{a}{a+b}\right| = \frac{1}{(a+b)(c+d)},$$

*from which we get*

$$\frac{1}{4\|C_I\|^2} \leq |d_I| \leq \frac{1}{\|C_I\|}.$$

*Hence the orbit under the* $\psi_I$ *of every* $w \in (0,1)$ *is dense in* $(0,1)$ *and therefore* $\dim_H R_{\boldsymbol{C}_2} = 1$*. Note that, since this IFS is not hyperbolic, Theorem 4 does not apply to it.*

*Let us get analytical bounds for* $s_{\mathcal{A}}$ *via* $\mu_{\boldsymbol{C}_2,\ell}$*. Unlike the previous example, the presence of parabolic elements in the semigroup does not allow to truncate the series to just a few terms because of its very slow convergence. Since* $\|C_i C_{i+1}^k\| = k+1$ *we get easily that*

$$\mu_{\boldsymbol{C}_2,0}(s) = \sum_{J\in\mathcal{J}^2} \|C_J\|^{-s} = 2\sum_{k=1}^{\infty} \|C_1 C_2^k\|^{-s} = 2\sum_{k=2}^{\infty} k^{-s} = 2(\zeta(s)-1),$$

*where* $\zeta(s)$ *is the Riemann's zeta function. The solution of* $\mu_{\boldsymbol{C}_2,0}(s) = 2^s$ *gives us the bound* $s_{\boldsymbol{C}_2} \geq 1.54$*. The first upper bound can be gotten from* $\mu_{\boldsymbol{C}_2,2}$*, obtained by replacing the two terms of norm 2 in* $\mu_{\boldsymbol{C}_2,0}(s)$ *, namely* $\|C_{12}\|^{-s}$ *and* $\|C_{21}\|^{-s}$*, with, respectively,* $\mu_{\boldsymbol{C}_2 A_{12}}(s)$ *and* $\mu_{\boldsymbol{C}_2 A_{21}}(s)$*. A direct calculation shows that*

$$\mu_{\boldsymbol{C}_2 A_{12}}(s) = \sum_{k=2}^{\infty}(2k+1)^{-s} + \sum_{k=4}^{\infty} k^{-s} = 2^{-s}\zeta(s,\frac{5}{2}) + \zeta(s) - 1 - 2^{-s} - 3^{-s},$$

*where* $\zeta(s,t)$ *is the Hurwitz zeta function, and by symmetry we know that* $\mu_{\boldsymbol{C}_2 A_{12}} = \mu_{\boldsymbol{C}_2 A_{21}}$*. Hence*

$$\mu_{\boldsymbol{C}_2,2}(s) = 2\left(2\zeta(s) + 2^{-s}\zeta(s,\frac{5}{2}) - 2 - 2^{1-s} - 3^{-s}\right)$$

*which gives the bounds* $1.7 \leq s_{\boldsymbol{C}_2} \leq 3.93$ *as solutions of* $\mu_{\boldsymbol{C}_2,2}(s) = 2^s$ *and* $\mu_{\boldsymbol{C}_2,2}(s) = 2^{-s}$*. A numerical evaluation of* $N_{\boldsymbol{C}_2}(k)$ *for* $k = 2^p$*,* $1 \leq p \leq 17$*, (taking about 1 hour of CPU time, see Table 3 for the evaluated values) gives* $s_{\boldsymbol{C}_2} = 2.0001$ *with a (heuristic) error of 1 on the last digit. This and the evaluation of* $\dim_H R_{\boldsymbol{C}_2}$ *above strongly suggest that* $s_{\boldsymbol{C}_2} = 2$*.*

It is interesting to consider the following generalization of the previous example, namely the free semigroups $\boldsymbol{C}_{2,\alpha} \subset SL_2(\mathbb{N})$ generated by

$$C_{1,\alpha} = \begin{pmatrix} \alpha & 0 \\ 1/\alpha & 1/\alpha \end{pmatrix}, C_{2,\alpha} = \begin{pmatrix} 1/\alpha & 1/\alpha \\ 0 & \alpha \end{pmatrix}.$$

In this case, in the same framework used above,

$$\psi_{1,\alpha}(\varphi) = \frac{\varphi}{\alpha^2 + (2-\alpha^2)\varphi}, \; \psi_{2,\alpha}(\varphi) = \frac{1 + (\alpha^2 - 1)\varphi}{2 + (\alpha^2 - 2)\varphi}.$$

A direct check shows that, for every fixed $\alpha \in (1, 2)$, the $\psi_{i,\alpha}$ are contractions on the invariant interval $[0, 1]$ and that they satisfy the open set condition with respect to it. Let $R_{\boldsymbol{C}_{2,\alpha}}$ be the limit set of the orbit of any point $w \in (0, 1)$ under the action induced by $\boldsymbol{C}_{2,\alpha}$. The very same argument used in the example above shows that $\dim_H R_{\boldsymbol{C}_{2,\alpha}} = 1$. As a corollary of Theorem 4 we get the following:

**Proposition 9.** $s_{\boldsymbol{C}_{2,\alpha}} = 2$ *for every* $\alpha \in (1, 2)$.

**Remark 3.** *The restriction on the possible values of* $\alpha$ *looks more like an artificial effect of a poor choice for the distance function rather than a true property of the semigroups. We believe that by choosing a ad-hoc metric and maybe slightly modifying the argument the proposition above can be extended to the half-line* $[1, \infty)$.

## 4.2 $n = 2, K = \mathbb{C}$

Now we consider the case of $2 \times 2$ *complex* matrices. The corresponding projective space is the *Riemann sphere* $\mathbb{C}P^1$, namely the complex plane with the addition of a point at infinity.

The geometry of Kleinian groups, namely of discrete subgroups of the Möbius group $PSL_2(\mathbb{C})$, is known to be extremely rich and is presumably even richer in case of Kleinian sub*semi*groups. A study of such semigroups in a general setting is therefore way beyond the scope of the present paper. Here we rather state first a somehow general property analogous to the real case above and then, as a source of examples, focus our attention on a particular but interesting case that we refer to as *complex projective Sierpinski gaskets.*

**Theorem 5.** *Let $f_i$ be $m$ linear automorphisms of $C^2$ represented in coordinates by the matrices $A_i \in SL_2(\mathbb{C})$ and let $\psi_i$ be the induced elements in $PSL_2(\mathbb{C})$. Assume that there exists some proper open set $V \subset \mathbb{C}P^1$ invariant by the $\psi_i$ and such that, for some affine chart $\varphi : \mathbb{C}P^1 \to \mathbb{C}$, the $\psi_i$ are contractions on $\varphi(\overline{V})$ with respect to the Euclidean distance and satisfy the "open set condition" $\psi_1(V) \cap \psi_2(V) = \emptyset$. Let $R_{\boldsymbol{A}} = \cap_{k=1}^{\infty} \left( \cup_{|I|=k} \psi_I(V) \right)$ be the corresponding residual set. Then $2 \dim_H R_{\boldsymbol{A}} = s_{\boldsymbol{A}}$.*

The proof of this theorem is almost verbatim the same of Theorem 4 and, correspondingly, we have the following corollary:

**Corollary 3.** *Under the hypotheses of the previous theorem, $2 \leq s_{\boldsymbol{A}} \leq 4$.*

**Definition 5.** *Let $f_1, f_2, f_3$ be volume-preserving linear automorphisms of $\mathbb{C}^2$ with real spectrum and denote by $A_1, A_2, A_3 \in SL_2(\mathbb{C})$ the corresponding matrices with respect to some coordinate system and by $\psi_1, \psi_2, \psi_3 \in PSL_2(\mathbb{C})$ their corresponding projective automorphisms acting on the Riemann sphere. Let $[e_i] \in \mathbb{C}P^1$ be a fixed point for $\psi_i$ corresponding to the largest eigenvalue $\lambda \geq 1$ of $f_i$. We say that the semigroup $\boldsymbol{F}$ generated by the $f_i$ (or, equivalently, the semigroup $\boldsymbol{A}$ generated by the $A_i$) is a* complex projective Sierpinski gasket *if the following conditions are satisfied:*

1. *$[f_i(e_j)] = [f_j(e_i)]$ for every $i, j$ with $i \neq j$;*
2. *the circle $\Gamma_k$ passing through $[e_i]$, $[e_j]$ (where $i, j, k$ is a permutation of $1, 2, 3$) and $[f_i(e_j)]$ is invariant under both $f_i$ and $f_j$, $i \neq j$;*
3. *$[f_k(e_i)]$ and $[f_k(e_j)]$ belong to the same connected component of $\mathbb{C}P^1 \setminus \Gamma_k$ for every permutation $(i, j, k)$ of $(1, 2, 3)$.*
4. *the circles $\Gamma_k$ do not intersect in the interior of the curvilinear triangle $T_{\boldsymbol{A}} \subset \mathbb{C}P^1$ having as vertices the $[e_i]$ and as sides the segments of the $\Gamma_k$ with vertices the points $[e_i]$ and $[e_j]$ containing the point $[f_i(e_j)]$.*

By construction every such gasket $\boldsymbol{A}$ is free and satisfies the open set condition with respect to the interior of $T_{\boldsymbol{A}}$. Since the Möbius group $PSL_2(\mathbb{C})$ is transitive on triples of distinct points, we assume without loss of generality in the rest of this section that $T_{\boldsymbol{A}}$ has vertices $[e_1] = [1 : 1]$, $[e_2] = [i : 1]$, $[e_3] = [-1 : 1]$ with respect to homogeneous coordinates $[z : w]$ and use the affine chart $w = 1$ with complex coordinate $z = x + iy$ for all calculations.

**Proposition 10.** *Let $f_1, f_2, f_3$ be volume-preserving linear automorphisms with real spectrum having respectively $e_1 = (1,1)$, $e_2 = (i,1)$, $e_3 = (-1,1)$ as eigenvectors corresponding to the largest eigenvalue and assume that*

$$\begin{cases} \psi_1([e_3]) = \psi_3([e_1]) = u + iv, \\ \psi_2([e_1]) = \psi_1([e_2]) = is, \\ \psi_3([e_2]) = \psi_2([e_3]) = -u + iv. \end{cases}$$

*A necessary condition for $f_1, f_2, f_3$ to generate a Sierpinski gasket* symmetric with respect to the imaginary axes*, namely such that $f_1(z) = \overline{f_2(-\bar{z})}$ and $f_3(z) = \overline{f_3(-\bar{z})}$, is that $\psi_1([e_3]) \in \Gamma$, where $\Gamma$ is the circle*

$$x^2 + y^2 - x(1 - s^2) - s^2 = 0. \tag{24}$$

*For $s = 0$ the condition is sufficient for $u \in [1/5, \alpha]$, where $\alpha \simeq 0.651$.*

*Proof.* A long but straightforward direct calculation shows that condition (24) is the only one coming from imposing that each one of the tetruples $[e_1]$, $[e_3]$, $\psi_1([e_3])$, $\psi_{11}([e_3])$ and $[e_1]$, $[e_2]$, $\psi_1([e_2])$, $\psi_{11}([e_2])$ identifies a single circumference. No further condition comes from $\psi_3$ and by symmetry we obtain an equivalent condition with respect to $\psi_2$.

When $s = 0$ another direct calculation shows that if $u < 1/5$ then $e_1$ is not anymore the eigenvector of $f_1$ corresponding to its largest eigenvalue. When $u = \alpha$ the circles $\Gamma_{13}$ and $\Gamma_{12}$ are tangent to each other and for $u > \alpha$ they intersect inside $T_{\boldsymbol{A}}$. □

**Example 12.** *Let us give a short survey of the kind of geometry we meet in case of complex projective Sierpinski gaskets symmetric with respect to the imaginary axes. For $u = 16/25 \simeq \alpha$ we get the gasket*

$$A_1 = \frac{1}{\sqrt{544}} \begin{pmatrix} 20 & 12i \\ -3i & 29 \end{pmatrix}, A_2 = \frac{1}{\sqrt{24}} \begin{pmatrix} 4 & 4 \\ 1 & 7 \end{pmatrix}, A_3 = \frac{1}{\sqrt{24}} \begin{pmatrix} 4 & -4 \\ -1 & 7 \end{pmatrix}.$$

*In Fig. 4 we show the orbit of a point under the action of the semigroup $\boldsymbol{A}$ generated by the $A_i$. The triangle $T_{\boldsymbol{A}}$ is convex and, correspondingly, the triangle $Z_{\boldsymbol{A}} = T_{\boldsymbol{A}} \setminus (\cup_{i=1}^3 T_{\boldsymbol{A}A_i})$ is concave. Each angle is almost zero because the sides of the triangle are almost tangent to each other, which corresponds to the fact that the limit value $\alpha$ is close to* 16/25*. The restriction to $T_{\boldsymbol{A}}$ of corresponding maps $\psi_i$ are contractive so Theorem 5 applies. A*

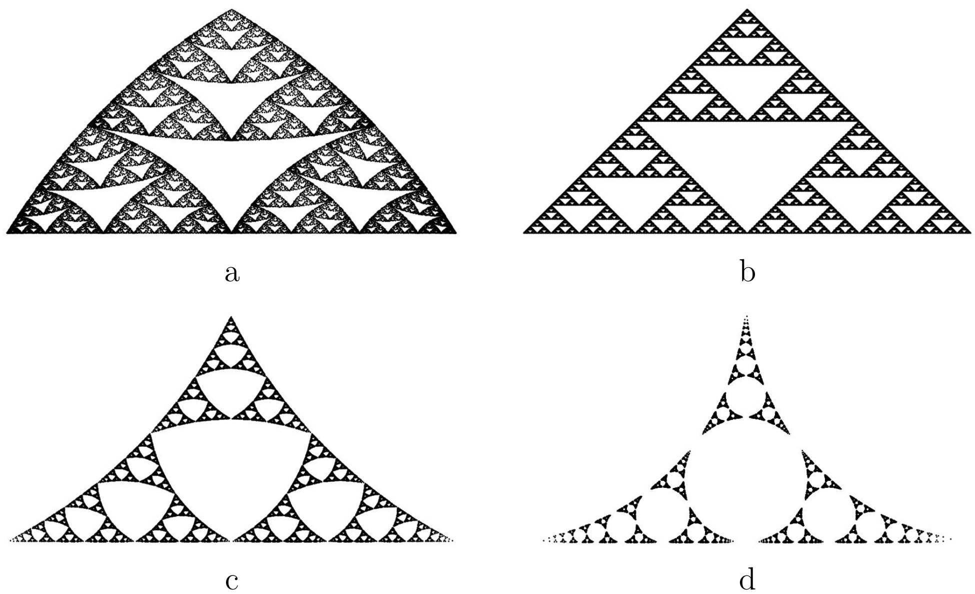


Figure 4: Limit sets of complex self-projective Sierpinski gaskets: (a) $u = 16/25$, $s_{\boldsymbol{A}} \simeq 2.88$, $\dim R_{\boldsymbol{A}} \simeq 1.44$; (b) $u = 1/2$, $s_{\boldsymbol{A}} = 2\log_2 3$, $\dim R_{\boldsymbol{A}} = \log_2 3$; (c) $u = 9/25$, $s_{\boldsymbol{A}} \simeq 2.88$, $\dim R_{\boldsymbol{A}} \simeq 1.44$; (d) $u = 1/5$, $s_{\boldsymbol{A}} \simeq 2.61$, $\dim R_{\boldsymbol{A}} \simeq 1.305$. In figure for each case we show the 19683 points of the orbit of a random point under the action of all matrices $A_I$ of the gasket with $|I| = 9$.

*rough numerical evaluation of the exponent of* $\boldsymbol{A}$ *gives* $s_{\boldsymbol{A}} \simeq 2.88$*, so that* $\dim R_{\boldsymbol{A}} \simeq 1.44$.

*By increasing u the curvature of the sides increases (we consider negative the curvature of concave sides) until it gets zero for* $u = 1/2$*. The semigroup is now generated by*

$$A_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ 0 & 2\end{pmatrix}, A_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}, A_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix}.$$

*In this case all sides are segments of straight lines and the gasket is diffeomorphic to the standard Sierpinski gasket in* $\mathbb{R}^2$*. It is easy to prove that* $s_{\boldsymbol{A}} = 2\log_2 3$ *and, correspondingly, we get the well-known result* $\dim R_{\boldsymbol{A}} = \log_2 3$*.*

*By increasing u further the curvature of the sides keeps increasing and therefore* $T_{\mathcal{A}}$ *becomes convex. For* $u = 9/25$ *the semigroup is generated by*

$$A_1 = \frac{1}{45}\begin{pmatrix}3 & 6i\\ 2i & 11\end{pmatrix}, A_2 = \frac{1}{\sqrt{24}}\begin{pmatrix}3 & 3\\ -1 & 7\end{pmatrix}, A_3 = \frac{1}{\sqrt{24}}\begin{pmatrix}3 & -3\\ 1 & 7\end{pmatrix}.$$

*The corresponding* $\psi_i$ *are contractive over* $T_{\mathcal{A}}$ *so that Theorem 5 still applies. A rough numerical evaluation of the exponent gives* $s_{\boldsymbol{A}} \simeq 2.88$ *so that* $\dim R_{\mathbb{C}} \simeq 1.44$*.*

*At the extremal value* $u = 1/5$ *every angle of the triangle is equal to* $\pi$*, namely every triangle* $Z_{\boldsymbol{A}}$ *is actually a circle. Indeed this Sierpinski gasket is actually the Apollonian gasket* $\boldsymbol{A}_3$*, introduced in the Motivational Example 2 and generated by*

$$A_1 = \begin{pmatrix}0 & i\\ i & 2\end{pmatrix}, A_2 = \frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 3\end{pmatrix}, A_3 = \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 3\end{pmatrix}.$$

*This time the corresponding* $\psi_i$ *are only non-expansive, which corresponds to the fact that* $\boldsymbol{A}_3$ *are parabolic.*

*Finally we point out that all these gaskets are fast. Here we outline the argument in case of the Apollonian gasket* $\boldsymbol{A}_3$ *but the same argument holds for all complex projective gaskets symmetric with respect to the imaginary axes. Note first of all that it is straightforward proving by induction that* $\|A_I\| = |(A_I)_{22}|$ *for every matrix* $A_I \in \boldsymbol{A}_3$*. Now consider the case*

$$A_I = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, A_{23} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 5\end{pmatrix}, A_L = \begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

*Then*

$$A_{23L} = \frac{1}{2}\begin{pmatrix} a+c & b+d \\ a+5c & b+5d \end{pmatrix},$$

*so that* $\|A_{I23L}\| = \frac{1}{2}|\gamma(b+d)+\delta(b+5d)| \geq 2|\delta||d| \geq \frac{1}{3}\|A_I\|\|A_{23L}\|$. *The case of* $A_{32}$ *is completely analogous to this. The remaining four combinations are instead analogous to the case of*

$$A_{12} = \frac{1}{2}\begin{pmatrix} -i & 3i \\ -2+i & 6+i \end{pmatrix}.$$

*This time*

$$A_{12L} = \frac{1}{2}\begin{pmatrix} -ia+3ic & -ib+3id \\ (-2+i)a+(6+i)c & (-2+i)b+(6+i)d \end{pmatrix}$$

*and* $\|A_{I12L}\| = \frac{1}{2}|\gamma(3id-ib)+\delta((i-2)b+(6+i)d)| \geq 2|\delta||d| \geq \frac{1}{5}\|A_I\|\|A_{12L}\|$. *Hence* $\boldsymbol{A}_3$ *is a fast gasket with coefficient not smaller than* $1/5$.

### 4.3 $n \geq 3, K = \mathbb{R}$

In the real case, projective maps induced by at least $3\times 3$ matrices are not conformal and we could find any simple way to prove analogues of Theorems 4 and 5. The non-triviality of the matter is granted by the well-known non-triviality of the theory of real self-affine sets (e.g. see [Fal88, FL98, ABVW10, FM11]). Indeed, since $PSL^{\pm}_n(\mathbb{R})$ contains a subgroup homeomorphic to the $(n-1)$-dimensional affine group, self-projective sets are at least as non-trivial as the self-affine ones (see Section 4.3.1 for more details).

Because of this and in order to provide motivation for the interest of real self-projective sets we restrict our attention to the following particular case:

**Definition 6.** *Let* $\boldsymbol{F} = \langle f_1,\dots,f_n\rangle$ *be a free semigroup of volume-preserving linear automorphisms of* $\mathbb{R}^n$ *and* $\psi_1,\dots,\psi_n \in PSL_n(\mathbb{R})$ *the induced projective automorphisms of* $\mathbb{R}P^{n-1}$. *Let* $\mathcal{E} = \{e_1,\dots,e_n\}$ *be a* $n$*-frame of* $\mathbb{R}^n$ *and* $\mathcal{E}^* = \{\varepsilon_1,\dots,\varepsilon_n\}$ *its dual frame. We say that* $\boldsymbol{F}$ *is a* real projective Sierpinski gasket *over* $\mathcal{E}$ *if the following conditions are satisfied:*

1. $f_i = A^k_{ij} e_k \otimes \varepsilon^j$ *with* $A^k_{ij} \geq 0$;
2. $f(e_i) = \lambda_i e_i$, *with* $\lambda_i = \max_{1\leq j\leq n}\{A^j_{ij}\}$;

*3.* $f_i(e_j) = \alpha e_i + \beta e_j$ *with* $\alpha, \beta > 0$*;*

*4.* $\psi_i([e_j]) = \psi_j([e_i])$*,* $i \neq j$*.*

*We say that* $\mathcal{E}$ *is a* proper frame *for* $\boldsymbol{F}$*. More generally given* $m < n$ *of the* $f_i$ *we say that they are a Sierpinski gasket if there exist automorphisms* $f_{m+1}, \ldots, f_n$ *such that* $\langle f_1, \ldots, f_n \rangle$ *is a Sierpinski gasket.*

Note that conditions 1–3 above imply that $span\{e_i\}$ is the only eigenspace of $f_i$ corresponding to its largest eigenvector, so that every proper frame for $\boldsymbol{F}$ identifies the same $n$ points $[e_i]$ on $\mathbb{R}\mathrm{P}^{n-1}$.

Denote by $C(\mathcal{E})$ the *positive cone* over $c$, namely the convex hull of the set $\cup_{i=1}^n \{\lambda e_i, \lambda > 0\}$. Then its projection on $\mathbb{R}\mathrm{P}^{n-1}$ is the same for every proper frame of $\boldsymbol{F}$ and we denote it by $T_{\boldsymbol{F}}$. This set is a $(n-1)$-simplex with the $n$ points $[e_i]$ as vertices. By points 2 and 3 of the definition above, $[e_i]$ is a fixed point for $\psi_i$ and each set $T_{\boldsymbol{F}f_i} \stackrel{\text{def}}{=} \psi_i(T_{\boldsymbol{F}})$ is a $(n-1)$-simplex having in common with every other $T_{\boldsymbol{F}f_j}$, $i \neq j$, the vertex $\psi_i([e_j])$. Like in case of the $(n-1)$-dimensional standard Sierpinski gasket in $\mathbb{R}^{n-1}$, the difference between $T_{\boldsymbol{F}}$ and $\cup_{i=1}^n T_{\boldsymbol{F}f_i}$ is the interior of a convex polyhedron with $n(n-1)/2$ vertices that we denote by $Z_{\boldsymbol{F}}$.

By repeating this procedure recursively we see that, at every step $k > 0$,

$$T_{k,\boldsymbol{F}} \stackrel{\text{def}}{=} \bigcup_{|I|=k} T_{\boldsymbol{F}f_I} = T_{\boldsymbol{F}} \setminus \left[ \bigcup_{|I|<k} Z_{\boldsymbol{F}f_I} \right]$$

It is standard to call $R_{\boldsymbol{F}} = \cap_{k \geq 0} T_{k,\boldsymbol{F}}$ the *residual set* of $\boldsymbol{F}$.

For sake of simplicity and conciseness we limit our discussion to the following subclass of Sierpinski gaskets:

**Definition 7.** *We say that a Sierpinski gasket* $\boldsymbol{F} = \langle f_1, \ldots, f_m \rangle$ *is* simple *when each* $f_i$ *either has only one eigenvalue (first kind) or has exactly two eigenvalues and the eigenspace corresponding to the larger one is 1-dimensional (second kind).*

**Example 13.** *The most important 1-parameter family of real projective Sierpinski gaskets we discuss in this paper is* $F_n^\alpha = \{f_1^\alpha, \ldots, f_n^\alpha\}$*,* $\alpha \geq 1$*,*

$$f_i^\alpha(e_j) = \alpha^{-\frac{1}{3}} \begin{cases} \alpha e_i, & i = j, \\ e_i + e_j, & i \neq j. \end{cases}$$

*For $n = 3$ the $f_i^\alpha$ are represented, with respect to any proper frame, by the matrices*

$$A_1^{\alpha,3} = \alpha^{-\frac{1}{3}} \begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_2^{\alpha,3} = \alpha^{-\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 1 & \alpha & 1 \\ 0 & 0 & 1 \end{pmatrix}, A_3^{\alpha,3} = \alpha^{-\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & \alpha \end{pmatrix}.$$

*As already shown in the introduction, for $\alpha = 1$ we get the cubic gasket $\boldsymbol{C}_3$ and for $\alpha = 2$ the (real projective generalization of the) standard Sierpinski gasket $\boldsymbol{S}_3$.*

Consider now the dual semigroup $\boldsymbol{F}^* \subset Aut((\mathbb{R}^n)^*)$ of a Sierpinski gasket.

**Proposition 11.** *Let $\boldsymbol{F}$ be a simple Sierpinski gasket over a $n$-frame $\mathcal{E} = \{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ generated by maps*

$$f_i(e_i) = \alpha_i e_i,\ f_i(e_j) = \beta_{ij} e_i + \gamma_i e_i.$$

*Then $\boldsymbol{F}^*$ is a simple Sierpinski gasket over the frame $\mathcal{H} = \{\eta^1, \dots, \eta^n\}$ of $(\mathbb{R}^n)^*$ defined by*

$$\eta^i = \sum_{j \neq i} \beta_{ij} \varepsilon^j + (\alpha_i - \gamma_i) \varepsilon^i. \tag{25}$$

*Proof.* By direct calculation we see that

$$f_i^*(\eta^i) = \sum_{j \neq i} \beta_{ij} f_i^*(\varepsilon^j) + (\alpha_i - \gamma_i) f_i^*(\varepsilon^i) =$$

$$= \sum_{j \neq i} \beta_{ij} \varepsilon^j + (\alpha_i - \gamma_i)(\alpha_i \varepsilon^i + \sum_{j \neq i} \beta_{ij} \varepsilon^j) =$$

$$= \alpha_i \left( \sum_{j \neq i} \beta_{ij} \varepsilon^j + (\alpha_i - \gamma_i) \varepsilon^i \right) = \alpha_i \eta^i$$

and

$$f_i^*(\eta^k) = \sum_{j \neq k} \beta_{kj} f_i^*(\varepsilon^j) + (\alpha_k - \gamma_k) f_i^*(\varepsilon^k) =$$

$$= \sum_{j \neq k,i} \beta_{kj} \varepsilon^j + \beta_{ki} \left( \sum_{j \neq i} \beta_{ij} \varepsilon^j + \alpha_i \varepsilon^i \right) + (\alpha_k - \gamma_k) \varepsilon^k = \beta_{ki} \eta^i + \gamma_i \eta^k.$$

□

**Lemma 3.** *Let* $\boldsymbol{F} = \langle f_i \rangle \subset Aut(V^n)$ *be a simple Sierpinski gasket over* $\mathcal{E} = \{e_i\}$*. Then the following inequalities hold:*

$$\begin{aligned} \|f_I\|_{\ell^1} &\le C \min_{\substack{1\le k,k'\le n \\ k\neq k'}} \{\|f_I(e_k)\|_{\ell^1} + \|f_I(e_{k'})\|_{\ell^1}\}, \qquad &&\textit{if } \boldsymbol{F} \textit{ is of the first kind.} \\ \|f_I\|_{\ell^1} &\le C \min_{1\le k\le n} \|f_I(e_k)\|_{\ell^1}, &&\textit{if } \boldsymbol{F} \textit{ is of the second kind,} \end{aligned} \tag{26}$$

*for some* $C > 0$*, where* $\|\omega\|_{\ell^1} = \sum_{1\le j\le n} |\omega_j|$*.*

*Proof.* Let $\mathcal{H} = \{\eta^i = \sum \hat\beta^i_j \varepsilon^j\}$ be the proper frame for $\boldsymbol{F}^*$ introduced in (25), where $\hat\beta^i_j = \beta_{ij} > 0$, $j \neq i$, and $\hat\beta^i_i = \alpha_i - \gamma_i \ge 0$. Clearly $\omega = \sum_{1\le i\le n} \eta^i \in C(\mathcal{H})$. By the previous proposition, $\omega_I = f^*_I(\omega) \in C(\mathcal{H})$ for all $I \in \mathcal{I}^n$. This means that $\omega_I = \sum_{1\le i\le n} (\omega_I)_i \varepsilon^i = \sum_{1\le i\le n} \lambda_i \eta^i$ (with $\lambda_i \ge 0$ and $\sum_{1\le i\le n} \lambda_i > 0$) so that $(\omega_I)_i = \sum_{1\le j\le n} \lambda_j \hat\beta^j_i$ and therefore

$$\|\omega_I\|_{\ell^1} = \sum_{1\le i\le n} (\omega_I)_i = \sum_{1\le i,j\le n} \lambda_j \hat\beta^j_i \le n \max_{1\le i,j\le n} \{\hat\beta^i_j\} \sum_{1\le i\le n} \lambda_i.$$

Now note that $(\omega_I)_k \ge (\min_{\hat\beta^i_k > 0} \hat\beta^i_k) \sum_{\hat\beta^i_k > 0} \lambda_i$ is always a non-empty condition.

If $\boldsymbol{F}$ is of the second kind then $\hat\beta^i_j > 0$ for all $i, j$, so that

$$\|\omega_I\| \le n \frac{\max\limits_{1\le i,j\le n} \{\hat\beta^i_j\}}{\min\limits_{1\le i\le n} \{\hat\beta^i_k\}} (\omega_I)_k$$

for all $1 \le k \le n$.

If $\boldsymbol{F}$ is of the first kind then $\hat\beta^i_k = 0$ iff $i = k$, namely every $\omega_k$ misses $\lambda_k$ in its expression, which we can recover by adding any other $\omega_{k'}$, $k' \neq k$. Hence in this case

$$\|\omega_I\| \le (n-1) \frac{\max\limits_{1\le i,j\le n} \{\hat\beta^i_j\}}{\min\limits_{\substack{1\le i,j\le n \\ i\neq j}} \{\hat\beta^i_j\}} ((\omega_I)_k + (\omega_I)_{k'})$$

for all $1 \le k, k' \le n$, $k \neq k'$.

Finally note that $f_I = \sum_{1\le i,j\le n} A^i_{Ij} e_i \otimes \varepsilon^j$, so that

$$\omega_I = f_I^*(\omega) = \sum_{1\le i,j,k\le n} \hat\beta^i_j A^j_{Ik} \varepsilon^k$$

and therefore

$$(\omega_I)_k \le \max_{1\le i,j\le n}\{\hat\beta^i_j\} \sum_{1\le j\le n} A^j_{Ik} = \max_{1\le i,j\le n}\{\hat\beta^i_j\} \|f_I^*(e_k)\|_{\ell^1}$$

and

$$(\min_{\hat\beta^i_j>0} \hat\beta^i_j)\|f_I\|_{\ell^1} = (\min_{\hat\beta^i_j>0} \hat\beta^i_j) \sum_{j,k} A^j_{Ik} \le \|\omega_I\|_{\ell^1}$$

from which follows the claim of this lemma. □

**Example 14.** *Consider the Sierpinski gaskets $F^{\alpha,n}$ introduced in Example 13. A proper frame for $(F^{\alpha,n})^*$ is given by $\eta^i = (\alpha-1)\varepsilon^i + \sum_{j\ne i}\varepsilon^j$, so that $\omega = \sum_{1\le i\le n}\eta^i = (\alpha+n-2)\sum_{1\le i\le n}\varepsilon^i$ and therefore*

$$\omega_I = f_I^*(\omega) = (\alpha+n-2)\sum_{1\le i\le n} f_I^*(\varepsilon^i) = (\alpha+n-2)\sum_{1\le i,k\le n} A^i_{Ik}\varepsilon^k =$$

$$= (\alpha+n-2)\sum_{1\le k\le n} \|f_I^*(e_k)\|_{\ell^1}\varepsilon^k.$$

*If $\alpha = 1$ then*

$$\|\omega_I\|_{\ell^1} = \sum_{i,j}\lambda_j\hat\beta^j_i = (n-1)\sum_j \lambda_j$$

*and*

$$(\omega_I)_i = \sum_{j\ne i}\lambda_j$$

*so that*

$$\|\omega_I\|_{\ell^1} \le (n-1)\left((\omega_I)_k + (\omega_I)_{k'}\right)$$

*or, equivalently,*

$$\|f_I\|_{\ell^1} \le (n-1)\left(\|f_I(e_k)\|_{\ell^1} + \|f(e_{k'})\|_{\ell^1}\right)$$

*for every $k \ne k'$*

*If $\alpha > 1$ then*

$$\|\omega_I\|_{\ell^1} = \sum_{i,j} \lambda_j \hat{\beta}_i^j = (n-1)\max\{\alpha - 1, 1\} \sum_j \lambda_j$$

*and*

$$(\omega_I)_i \geq \min\{\alpha - 1, 1\} \sum_i \lambda_j$$

*so that*

$$\|\omega_I\|_{\ell^1} \leq (n-1)\min\{\alpha - 1, 1\}(\omega_I),$$

*or, equivalently,*

$$\|f_I\|_{\ell^1} \leq (n-1)\min\{\alpha - 1, 1\}\|f_I(e_k)\|_{\ell^1},$$

*for all $k$.*

**Proposition 12.** *Every simple Sierpinski gasket is a fast gasket.*

*Proof.* The key fact here is that in every Sierpinski gasket $\boldsymbol{F}$ with a proper frame $\mathcal{E}$, for any $i \neq j$ and every $k$, $f_{ij}(e_k)$ is linearly dependent on both $e_i$ and $e_j$. Indeed if $k \neq j$ then

$$f_{ij}(e_k) = f_i(\beta_{jk}e_j + \alpha_{jk}e_k) = \alpha_{jk}f_i(e_k) + \beta_{jk}\left(\beta_{ij}e_i + \alpha_{ij}e_j\right),$$

while if $k = j$ then

$$f_{ij}(e_j) = \alpha_{jj}(\beta_{ij}e_i + \alpha_{ij}e_j).$$

Hence the matrix representing $f_{ij}$ with respect to $\mathcal{E}$ has at least (actually, exactly) two rows with all non-zero coefficients. This means that every column of the matrix representing $f_{ijL}$ is a linear combination with strictly positive coefficients of $f_L(e_i)$ and $f_L(e_j)$ with possibly some other positive contribution from the other vectors.

From this we deduce immediately that

$$\|f_{IijL}\|_{\ell^1} \geq C\|f_I\|_{\ell^1}\left(\|f_L(e_i)\|_{\ell^1} + \|f_L(e_j)\|_{\ell^1}\right)$$

for some $C \geq 0$ and therefore, by Lemma 3, that

$$\|f_{IijL}\|_{\ell^1} \geq C'\|f_I\|_{\ell^1}\|f_L\|_{\ell^1}$$

for some $C' \geq 0$. Since $\|f_L\|_{\ell^1} \geq C''\|f_{ijL}\|_{\ell^1}$ for all $i \neq j$ and some $C'' > 0$, our claim follows. $\square$

**Proposition 13.** *Let $\boldsymbol{F} \subset Aut(\mathbb{R}^n)$ be a simple Sierpinski gasket, $\mathcal{E}$ a proper frame of $\boldsymbol{F}$ and $\mu$ any measure of $\mathbb{R}P^{n-1}$ in the measure class of the* round measure, *namely the measure induced by the metric of sectional curvature identically equal to 1. Then there exist constants $A, B, C, D > 0$ s.t.*

$$\frac{A}{\|f_I\|^n} \le \mu(T_{\boldsymbol{F} f_I}) \le \frac{B}{\|f_I\|^{a_n}}, \quad \frac{C}{\|f_I\|^n} \le \mu(Z_{\boldsymbol{F} f_I}) \le \frac{D}{\|f_I\|^n},$$

*where $a_n = n$ if $\boldsymbol{F}$ is of the second kind and $a_n = n-1$ if it is of the first kind.*

*Proof.* It is enough to prove the claim in some chart containing $T_{\boldsymbol{F}}$. We fix coordinates $(x^1, \dots, x^n)$ so that the vectors of $\mathcal{E}$ are

$$e_1 = (1, 0, \dots, 0, 1), \dots, e_{n-1} = (0, \dots, 0, 1, 1), e_n = (0, \dots, 0, 1)$$

and use the chart $x^n = 1$. In this chart we pick any smooth measure $\nu$ of finite total volume and with constant density equal to 1 within $T_{\boldsymbol{F}}$.

Note that $f_I = A^i_{Ik} e_i \otimes \varepsilon^k = A^i_{Ik} e^j_i \partial_j \otimes \varepsilon^k$, where the last row of the matrix $A^i_{Ik} e^j_i$ contain the $\ell^1$ norms of the vectors $f_I(e_i)$. A direct calculation shows that

$$\mu(T_{\boldsymbol{F} f_I}) = \frac{1}{n! \prod_{k=1}^{n} A^i_{Ik} e^n_i} = \frac{1}{n! \prod_{k=1}^{n} \|f_I(e_k)\|_{\ell^1}}.$$

Clearly $\dfrac{\prod_{k=1}^{n} \|f_I(e_k)\|_{\ell^1}}{\|f_I\|^n_{\ell^1}} \le 1$. By Proposition 3, if $\boldsymbol{F}$ is of the second kind then

$$A \le \frac{\prod_{k=1}^{n} \|f_I(e_k)\|_{\ell^1}}{\|f_I\|^n_{\ell^1}}$$

for some $A > 0$. If it is of the first kind assume, for the argument sake, that $\|f_I(e_1)\|_{\ell^1} \le \dots \|f_I(e_n)\|$. Then

$$\frac{\|f_I\|^{n-1}_{\ell^1}}{\prod_{k=1}^{n} \|f_I(e_k)\|_{\ell^1}} = \frac{(\sum_{k=1}^{n} \|f_I(e_k)\|_{\ell^1})^{n-1}}{\prod_{k=1}^{n} \|f_I(e_k)\|_{\ell^1}} \le$$

$$\leq C \prod_{k=1,n-1} \frac{\|f_I(e_k)\|_{\ell^1} + \|f_I(e_{k+1})\|_{\ell^1}}{\|f_I(e_{k+1})\|_{\ell^1}} \leq 2^{n-1}C$$

The geometry of $Z_{\boldsymbol{F}f_I}$ is more complex. We divide it in $n$ $(n-1)$-simplices $Z_i$, where $Z_i$'s vertices are the $(n-1)$ points $[f_i(f_I(e_j))]$, $j \neq i$, plus the point $[\sum_{1\leq i\leq n} f_I(e_i)]$. Then

$$\mu(Z_i) = \frac{1}{n!\|f_I(\sum_{1\leq i\leq n} e_i)\|_{\ell^1} \prod_{j\neq i} \|\beta_{ij}(e_I)_i + \gamma_i(e_I)_j\|_{\ell^1}} = \frac{1}{n!\|f_I\|_{\ell^1} \prod_{j\neq i} \|\beta_{ij}(e_I)_i + \gamma_i(e_I)_j\|_{\ell^1}}$$

since all components of all vectors are positive. Even in this case then

$$\frac{1}{\mu(Z_i)\|f_I\|^n_{\ell^1}} \leq n!$$

and

$$\frac{\|f_I\|^n_{\ell^1}}{\|f_I\|_{\ell^1} \prod_{j\neq i} \|\beta_{ij}(e_I)_i + \gamma_i(e_I)_j\|_{\ell^1}} \leq \frac{\|f_I\|^{n-1}_{\ell^1}}{\max\{\beta,\gamma\} \prod_{j\neq i} \|f_I(e_i) + f(e_j)\|_{\ell^1}} \leq A$$

for some $A$. $\square$

Next proposition supports the idea that the residual set of a Sierpinski gasket, or at least of a simple one, have non-integer dimension:

**Proposition 14.** *The residual set of a simple Sierpinski gasket $\boldsymbol{F} \subset SL^{\pm}_n(\mathbb{R})$ is a null set with respect to the measure class of the round measure on $\mathbb{R}P^{n-1}$.*

*Proof.* From the previous proposition we see that

$$1 \leq \frac{\mu(T_{\boldsymbol{F}f_I})}{\mu(Z_{\boldsymbol{F}f_I})} \leq \frac{C}{\|f_I\|}$$

if $\boldsymbol{F}$ is of the first kind and

$$1 \leq \frac{\mu(T_{\boldsymbol{F}f_I})}{\mu(Z_{\boldsymbol{F}f_I})} \leq C$$

if it is of the second. Let us first assume that $\boldsymbol{F}$ is of the second kind and let

$$S_k = \sum_{|I|=k+1} \mu(T_{\boldsymbol{F}f_I}) \text{ and } P_k = \sum_{|I|=k} \mu(Z_{\boldsymbol{F}f_I}).$$

Then

$$S_{k+1} = S_k - P_k \le S_k(1 - C)$$

and therefore

$$S_k - S_{k+1} \ge C S_k$$

so that, after making a telescopic sum, we get

$$S_1 - \lim_{k\to\infty} S_k \ge C \lim_{k\to\infty} k S_k$$

which immediately implies that $\lim_{k\to\infty} S_k = 0$.

If $\boldsymbol{F}$ is of the first kind then $\mu(T_{\boldsymbol{F}f_I})/\mu(Z_{\boldsymbol{F}f_I}) \le C/\min_{|J|=|I|} \|f_J\|$. It is easy to check that $\min_{|J|=|I|} \|f_J\|$ is proportional to $|I|$ and therefore

$$S_{k+1} = S_k - P_k \le S_k(1 - C/k)$$

and

$$S_k - S_{k+1} \ge C S_k / k$$

so that

$$S_1 - \lim_{k\to\infty} S_k \ge C \lim_{k\to\infty} S_k \sum_{1\le j\le k} 1/j.$$

Since the series $1/j$ diverges we get again that $\lim_{k\to\infty} S_k = 0$. $\square$

### 4.3.1 Affine Sierpinski Gaskets

In [FL98] Falconer and Lammering studied in detail the family of affine Sierpinski gaskets $S_{a,b}$, $a, b \in (0,1)$, defined by the affine transformations

$$\begin{aligned}
S_1 \begin{pmatrix} x \\ y \end{pmatrix} &= (1-a) \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ a \end{pmatrix} \\
S_2 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
S_3 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1-b & 1-a-b \\ 0 & a \end{pmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix}.
\end{aligned}$$

Notice that the case $a = \frac{1}{2}, b = \frac{1}{2}$ corresponds to the standard Sierpinski gasket. In particular they proved that the box dimension of the corresponding residual set $R_{a,b}$ is given by the unique root of the equation

$$(1-a)^s + ab^{s-1} + a(1-b)^{s-1} = 1 \tag{27}$$

in the triangle $T_1 = \{(a, b) \in (0, 1)^2, a \geq \max\{b, 1 - b\}\}$ and of

$$(1 - a)^s + a^{s-1} = 1 \tag{28}$$

in the opposite triangle $T_2 = \{(a, b) \in (0, 1)^2, a \leq \min\{b, 1-b\}\}$. This setting provides a convenient source of examples for comparing the box dimension of the residual set of a real projective Sierpinski gasket with the corresponding gasket exponent. Indeed the injection sending the affine transformation $S(x) = Tx + v$, with $T \in M_2(\mathbb{R})$ and $x, v \in \mathbb{R}^2$, into the $3 \times 3$ matrices $S \to \begin{pmatrix} T & v \\ 0 & 1 \end{pmatrix}$ applied to the $S_i$ gives the three matrices

$$M_1 = \begin{pmatrix} 1-a & 0 & 0 \\ 0 & 1-a & a \\ 0 & 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_3 = \begin{pmatrix} 1-b & 1-a-b & b \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that the action induced by the $A_i$ on $\mathbb{R}P^2$ in the affine chart $z = 1$ coincides with the action of the $S_i$.

Like in Example 9, these matrices are upper triangular and $\|M_I\| = 1$ for all $I \in \mathcal{I}^3$. To see this first of all we let $M = \begin{pmatrix} \alpha & \lambda & \mu \\ 0 & \beta & \nu \\ 0 & 0 & 1 \end{pmatrix}$ and notice that $0 \leq \lambda + \beta \leq 1$ and $0 \leq \mu + \nu \leq 1$. Indeed we can limit the discussion to left multiplication by $M_1$ and $M_3$ and it is immediate to verify by induction that, assuming the inequalities above for $M_I$, the products

$$M_1 M = \begin{pmatrix} (1-a)\alpha & (1-a)\lambda & (1-a)\mu \\ 0 & (1-a)\beta & (1-a)\nu + \alpha \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$M_3 M = \begin{pmatrix} (1-b)\alpha & (1-b)\lambda + (1-a-b)\beta & (1-b)\mu + (1-a-b)\nu + b \\ 0 & a\beta & a\nu \\ 0 & 0 & 1 \end{pmatrix},$$

satisfy the same inequalities. Now consider the semigroup $\boldsymbol{A}_{a,b}$ generated by the $A_i = M_i / \det M_i^{1/3} \in SL_3(\mathbb{R})$. Clearly $\|A_I\| = \det M_{i_1}^{-1/3} \cdots \det M_{i_k}^{-1/3}$ for every $I = i_1 \cdots i_k$ and therefore

$$\zeta_{\boldsymbol{A}_{a,b},k} = \sum_{I \in \mathcal{I}^3_k} \|A_I\|^{-s} = (\det M_1^{s/3} + \det M_2^{s/3} + \det M_3^{s/3})^k.$$

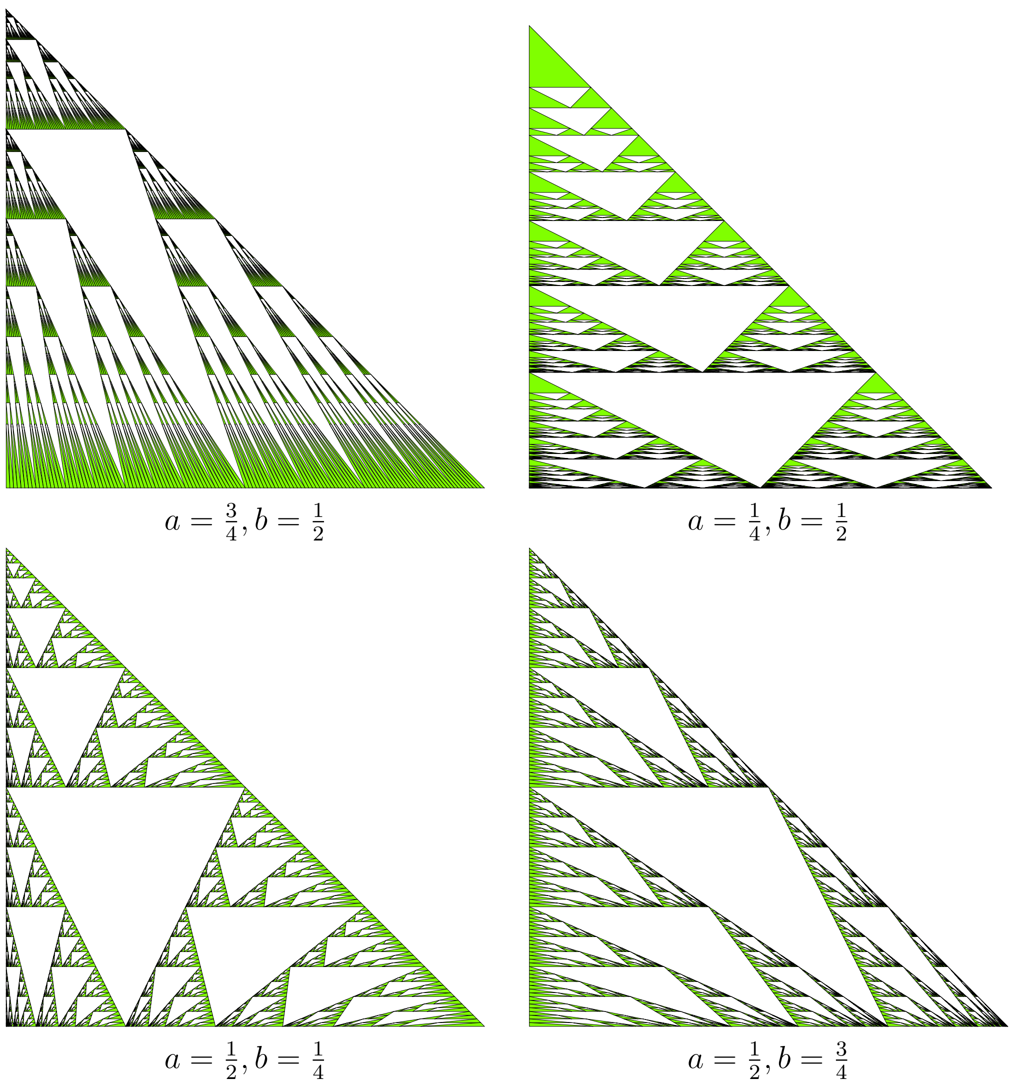


Figure 5: Affine Sierpinski gaskets $\boldsymbol{A}_{a,b}$ for four possible pairs $a, b$. For each one we plot (in green) the set $T_{7,\boldsymbol{A}_{a,b}}$. For the upper two the box dimensions can be evaluated analytically and their first six digits are, respectively, $\dim_B R_{\frac{3}{4},\frac{1}{2}} = 1.72368$ and $\dim_B R_{\frac{1}{4},\frac{1}{2}} = 1.68886$. The numerical evaluation of the box dimensions with an elementary box-counting algorithm gives: $\dim_B R_{\frac{3}{4},\frac{1}{2}} \simeq 1.71$, $\dim_B R_{\frac{1}{4},\frac{1}{2}} \simeq 1.66$, $\dim_B R_{\frac{1}{2},\frac{1}{4}} \simeq 1.60$, $\dim_B R_{\frac{1}{2},\frac{3}{4}} \simeq 1.60$. See Table 1 for a comparison of the box dimension of these and other affine gaskets with the expoentn of the corresponding real projective Sierpinski gaskets.

By Theorem 2 then the exponent $s_{\boldsymbol{A}_{a,b}}$ is the unique solution of the equation

$$(1-a)^{2s/3} + (ab)^{s/3} + (a(1-b))^{s/3} = 1. \tag{29}$$

**Proposition 15.** $2s_{\boldsymbol{A}_{a,b}} \leq 3\dim_B R_{a,b}$ *within the two triangles* $T_{1,2}$*, with the equal sign holding only in their common vertex.*

*Proof.* After writing (29) in terms of $t = 2s/3$ and renaming $t$ to $s$ we are left with the equation $(1-a)^s + (ab)^{s/2} + (a(1-b))^{s/2} = 1$. Comparing this expression with the left-hand sides of (27) and (28) we see that it is enough to prove that

$$(ab)^{s/2} + (a(1-b))^{s/2} \leq \min\{a^{s-1}, ab^{s-1} + a(1-b)^{s-1}\}.$$

Since for obvious geometrical reasons $\dim_B R_{a,b} \leq 2$ we can assume in the following $s \geq 2$. Let us denote respectively by $f_{a,b}(s), g_a(s), h_{a,b}(s)$ the three functions above and notice that, since by hypothesis $a, b, 1-b \in (0,1)$, they are all strictly monotonically decreasing functions of $s$ converging to 0 as $s \to \infty$. Moreover $f_{a,b}(2) = g_a(2) = h_{a,b}(2) = a$ and since these functions can have only one intersection it is enough to verify their behaviour for $s \to 0$. A direct calculation shows that, for every pair $a, b \in (0,1)^2$, we have that $\lim_{s\to 0} f_{a,b}(s) = 2$ while $\lim_{s\to 0} g_a(s) = \lim_{s\to 0} h_a(s) = \infty$. $\square$

Numerical experiments (see Fig. 5) clearly suggest that, even outside of the triangles $T_{1,2}$, $\dim_B R_{a,b}$ always larger than $\frac{2}{3}s_{\boldsymbol{A}_{a,b}}$ with the only exception of the case $a = 1/2, b = 1/2$, when these two quantities coincide. Moreover it appears that, roughly, $\frac{2}{3}s_{\boldsymbol{A}_{a,b}} \geq \frac{9}{10}\dim_B R_{\boldsymbol{A}}$.

### 4.3.2 The cubic semigroups $\boldsymbol{C}_n^\alpha$

Recall that, by definition, $f_i^\alpha(e_i) = \alpha e_i$, $f_i^\alpha(e_j) = e_j + e_i$, $j \neq i$.

$\boldsymbol{n = 3}$. In $\mathbb{R}^3$ we use coordinates $(x, y, z)$ with respect to the frame $e_1' = e_1 + e_3$, $e_2' = e_2 + e_3$, $e_3' = e_3$, so that the $f_i$ are represented by the matrices

$$A_1 = \begin{pmatrix} \alpha-1 & 0 & 1 \\ 0 & 1 & 0 \\ \alpha-2 & 0 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha-1 & 1 \\ 0 & \alpha-2 & 2 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2-\alpha & 2-\alpha & \alpha \end{pmatrix}.$$

In the affine chart $[x:y:z] \to (u,v) = (x/z, y/z)$ of $\mathbb{R}P^2$ we have therefore

| $a$ | $b$ | $s_{\boldsymbol{A}}$ (num.) | $s_{\boldsymbol{A}}$ (anal.) | $2s_{\boldsymbol{A}}/3$ | $\dim_B R_{\boldsymbol{A}}$ (num.) | $\dim_B R_{\boldsymbol{A}}$ (anal.) |
|---|---|---|---|---|---|---|
| 1/4 | 1/2 | 2.44 | 2.42632 | 1.61755 | 1.66 | 1.68886 |
| 3/4 | 1/2 | 2.48 | 2.45425 | 1.63617 | 1.71 | 1.72368 |
| 1/2 | 3/4 | 2.35 | 2.34443 | 1.56295 | 1.60 | – |
| 1/5 | 3/10 | 2.44 | 2.43735 | 1.62490 | 1.76 | 1.71262 |
| 4/5 | 3/10 | 2.47 | 2.46960 | 1.64640 | 1.77 | – |
| 3/10 | 1/5 | 2.43 | 2.37354 | 1.58236 | 1.75 | – |
| 7/10 | 1/5 | 2.35 | 2.35249 | 1.56833 | 1.72 | 1.63373 |

Table 1: Values of the exponents of affine Sierpinski gaskets $\boldsymbol{A}_{a,b}$ for several pairs $a, b$ and of the box dimension of the corresponding residual sets $R_{a,b}$. Numerical evaluations for $s_{\boldsymbol{A}}$ were done by calculating $N_{\boldsymbol{A}}(k)$ for the values $k = 2^p$, $p = 1, \dots, 12$, and are presented to motivate our confidence in a relative error not bigger than 1% in the other evaluations provided throughout the paper when an analytical evaluation is not available. Numerical evaluations for the box dimension of $R_{a,b}$ were done via an elementary box-counting algorithm and a comparison with the available analytical evaluations suggest that their relative error is about 10%.

that

$$\begin{cases} \psi_1(x,y) = \left(\frac{(\alpha-1)u+1}{(\alpha-2)u+2}, \frac{v}{(\alpha-2)u+2}\right) \\ \psi_2(x,y) = \left(\frac{u}{(\alpha-2)v+2}, \frac{(\alpha-1)v+1}{(\alpha-2)v+2}\right) \\ \psi_3(x,y) = \left(\frac{u}{(2-\alpha)(u+v)+\alpha}, \frac{v}{(2-\alpha)(u+v)+\alpha}\right) \end{cases}$$

and the vertices of the invariant triangle $T_{F_3^\alpha}$ are $[e_1] = (1,0)$, $[e_2] = (0,1)$ and $[e_3] = (0,0)$. A direct calculation of the eigenvalues of the Jacobian matrices $D\psi_i$ shows that, within $T_{F_3^\alpha}$,

$$\min\{\frac{1}{\alpha}, \frac{\alpha}{4}\} d(x,y) \leq d(\psi_i(x), \psi_i(y)) \leq \max\{\frac{1}{\alpha}, \frac{\alpha}{4}\} d(x,y)$$

for all $i = 1,2,3$, namely the semigroup $\langle\psi_1, \psi_2, \psi_3\rangle$, *as a IFS*, is hyperbolic for $\alpha \in (1,4)$ and parabolic for $\alpha = 1, 4$. The $\psi_i$ are not contractions *with respect to the Euclidean distance in this chart* for the other values of $\alpha$ (see Fig. 6 for the plot of $T_{7,F_3^\alpha}$ for several values of $\alpha$).

Analytical bounds for the Hausdorff dimension of the residual sets $R_{F_3^\alpha}$

| $\alpha$ | $s_{F_3^\alpha}$ | $2s_{F_3^\alpha}/3$ | $\dim_B R_{F_3^\alpha}$ |
|---|---|---|---|
| 1 | 2.447 | 1.631 | 1.72 |
| 1.3 | 2.395 | 1.596 | 1.72 |
| 1.7 | 2.377 | 1.585 | 1.71 |
| 2 | 2.359 | 1.573 | 1.59 |
| 3 | 2.378 | 1.586 | 1.71 |
| 4 | 2.389 | 1.593 | 1.73 |
| 7 | 2.394 | 1.596 | 1.76 |

Table 2: Numerical evaluation of the exponent of the real projective gaskets $F_3^\alpha$ and of the box dimension of the corresponding residual sets for several values of $\alpha$. No analytical formula is known for these quantity. These data confirms the relation $2s_{\boldsymbol{A}}/3 \leq \dim_B R_{\boldsymbol{A}}$ already observed in Table 1 and the fact that roughly $2s_{\boldsymbol{A}}/3 \geq 9\dim_B R_{\boldsymbol{A}}/10$.

can be obtained via Propositions 9.6 and 9.7 in [Fal90], namely

$$\min\{\log_3 \frac{4}{\alpha}, \frac{1}{\log_3 \alpha}\} \leq \dim_H R_{F_3^\alpha} \leq \max\{\log_3 \frac{4}{\alpha}, \frac{1}{\log_3 \alpha}\}.$$

For $\alpha = 2$ we get, as expected, $\dim_H R_{F_3^\alpha} = \log_3 2$.

Analytical bounds for the exponents $s_{\boldsymbol{F}_3^\alpha}$ can be obtained from Theorem 1. Here we present calculations for $F_3^1 = \mathcal{C}_3$, the cubic gasket. Due to the symmetry between the generators it turns out that

$$\mu_{\mathcal{C}_3}(s) = 6\mu_{\mathcal{C}_3 A_{12}} = 3 \cdot 2^{1-s}\zeta(s),$$

from which, as the unique solution of $\mu_{\mathcal{C}_3}(s) = 3^s$, we get the lower bound $1.52 \leq s_{\mathcal{C}_3}$. To get the first upper bound we must consider the function

$$\mu_{\mathcal{C}_3,2}(s) = 3 \cdot 2^{1-s}\left(3\zeta(s) + 2^{2-s}\zeta(s, \frac{7}{4}) - 2^{1-s} - 3\right),$$

from which we get $1.7 \leq s_{\mathcal{C}_3} \leq 7.1$ as the unique solutions of $\mu_{\mathcal{C}_3,2}(s) = 3^{\pm s}$. In order to get more meaningful bounds we should consider some $\mu_{\mathcal{C}_3,k}$ with a large $k$ but leave this to a future paper. Interpolating on the curve $\log N_{\mathcal{C}_3}(k)$ as function of $\log k$ for $k = 2^r$, $1 \leq r \leq 13$, we get a reliable estimate of $s_{\mathcal{C}_3} \simeq 2.444$. A rough numerical evaluation of the box dimension of $R_{\mathcal{C}_3}$ by counting the number of squares needed to cover the fractal gives $\dim_B \simeq 1.72$, compatible with the relation $3\dim_B R_{\mathcal{C}_3} \geq 2s_{\mathcal{C}_3}$ suggested in Conjecture 1.

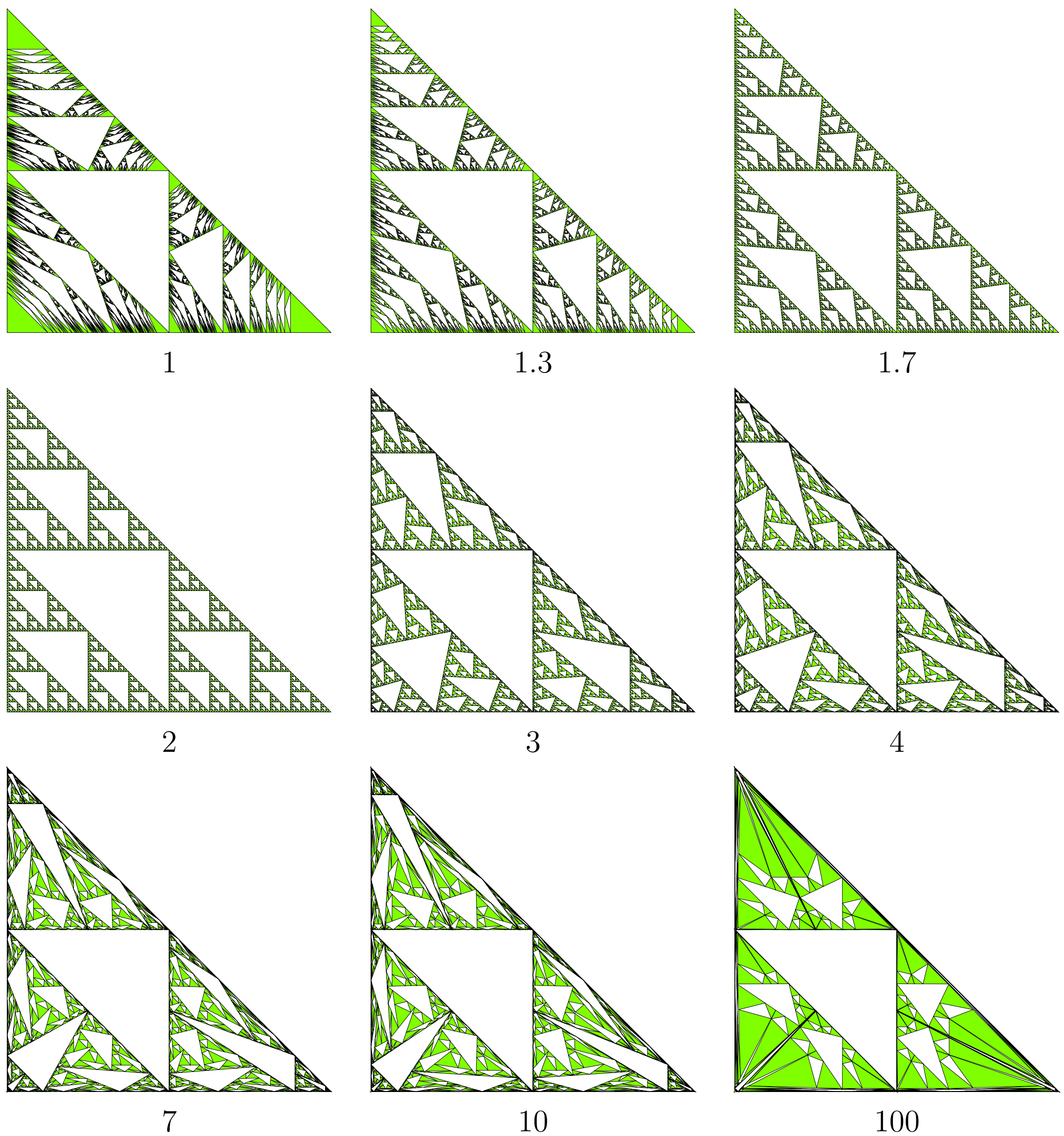


Figure 6: Real projective Sierpinski gaskets $F_3^\alpha$ for several values of $\alpha$. For each one we plot (in green) the set $T_{7,F_3^\alpha}$. Heuristic numerical estimates of their exponents and of the box dimension for the corresponding residual sets for $\alpha \leq 7$ are given in Table 2

$\boldsymbol{n \geq 4.}$ In $\mathbb{R}^n$ we use coordinates $(x^1, \dots, x^n)$ with respect to the frame $e'_1 = e_1 + e_n, \dots, e'_{n-1} = e_{n-1} + e_n, e'_n = e_n$. For $n = 4$ the matrices $A_1$ and $A_4$ are given by

$$A_1 = \begin{pmatrix} \alpha-1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha-2 & 0 & 0 & 2 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2-\alpha & 2-\alpha & 2-\alpha & \alpha \end{pmatrix}$$

and $A_2$ and $A_3$ can be obtained via permutations of $A_1$. Similarly happens for $n \geq 4$. Correspondingly we use coordinates $u^i = x^i/x^n$, $i = 1, \dots, n-1$, and obtain

$$\psi_1(u^i) = \left( \frac{(\alpha-1)u^1 + 2}{(\alpha-1)u^1 + 2}, \frac{u^2}{(\alpha-1)u^1 + 2}, \dots, \frac{u^{n-1}}{(\alpha-1)u^1 + 2} \right),$$

similarly for $i < n-1$ and

$$\psi_{n-1}(u^i) = \left( \frac{u^1}{(2-\alpha)(u^1 + \dots + u^{n-1}) + \alpha}, \dots, \frac{u^{n-1}}{(2-\alpha)(u^1 + \dots + u^{n-1}) + \alpha} \right).$$

A direct evaluation of the eigenvalues of the Jacobian matrices of the $\psi_i$ gives the same result we got for $n = 3$. In particular for every $n \geq 3$ we have that the gasket $\boldsymbol{F}^\alpha_n$ is a hyperbolic IFS for $\alpha \in (1, 4)$ and a parabolic IFS for $\alpha = 1, 4$. The bounds on the Hausdorff dimension of the residual sets give

$$\min\{\log_n \frac{4}{\alpha}, \frac{1}{\log_n \alpha}\} \leq \dim_H R_{F^\alpha_n} \leq \max\{\log_n \frac{4}{\alpha}, \frac{1}{\log_n \alpha}\}.$$

For $n = 4, \alpha = 2$ we get the well-known result that the dimension of the standard Sierpinski tetrahedron is equal to 2. Numerical evaluations suggest that the same could hold for the 4-dimensional version of the cubic gasket (see Fig. 7 for a picture of the two sets).

### 4.3.3 The Apollonian gasket

We conclude the paper with a brief discussion on the Apollonian semigroup, namely the semigroup $\boldsymbol{H} \subset SL_4(\mathbb{N})$ generated by the matrices $H_1, H_2, H_3$ introduced in the Motivational Example 2. This case was thoroughly studied, somehow implicitly, by Boyd, in particular in [Boy72, Boy73b, Boy82], in the context of the sequence of curvatures in an Apollonian gasket and

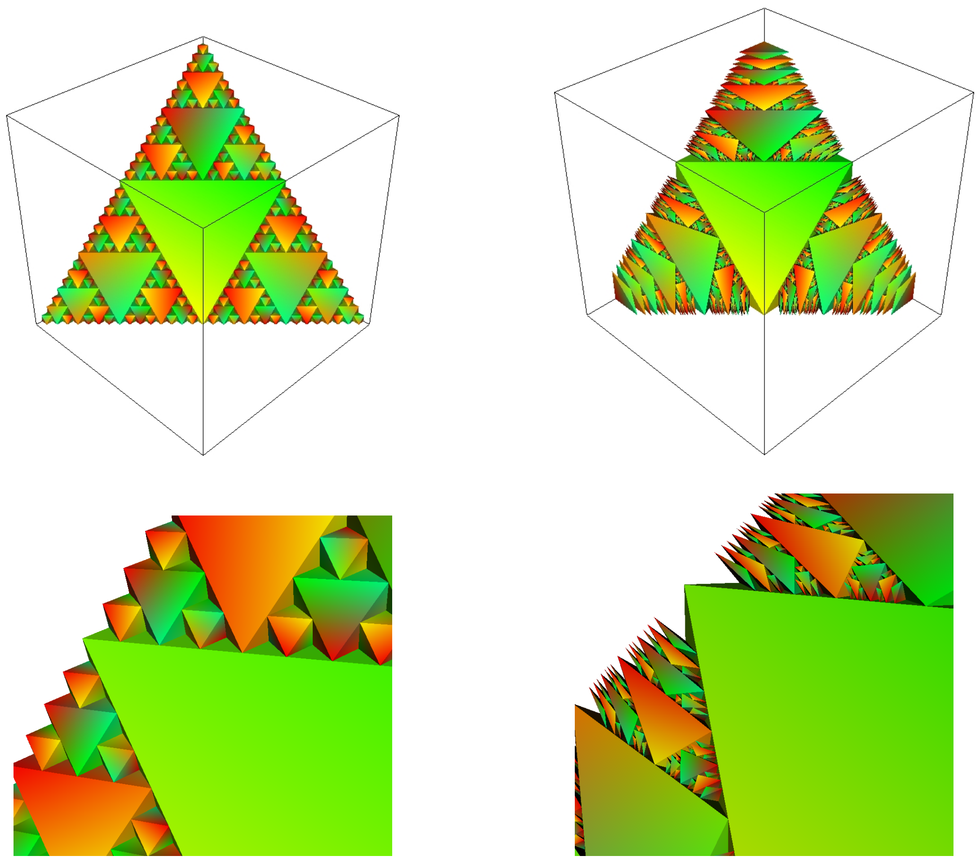

Figure 7: Images of the Sierpinski ($F_4^2$) and the cubic ($F_4^1$) tetrahedra. In figure we show a full picture (above) and a detail (below) for the sets $T_{5,F_4^2}$ (left) and $T_{5,F_4^1}$ (right).

Boyd's investigation and arguments were the archetype for most results and arguments in Section 3 of the present paper.

Recall that the matrix $H_1$ has only one eigenvalue (hence equal to 1) and therefore, even though $H_2$ and $H_3$ have eigenvalues larger than 1, $\boldsymbol{H}$ is a *parabolic* gasket. Next proposition grants that our results do apply in fact, as expected, to $\boldsymbol{H}$ itself and shows that $s_{\boldsymbol{H}} \leq \infty$ with arguments that entail only the matrices themselves.

**Lemma 4.** *Assume that matrices $A_1, \dots, A_m \in M_n(\mathbb{N})$ have the following properties:*

1. *they have some number $k \neq 1$ of rows containing a single entry equal to 1 and all other equal to 0 and these entries equal to 1 belong all to different columns and in those columns all entries are either 0 or 1;*

2. *other rows are such that each of their entries is smaller than the sum of the remaining $n-1$ entries.*

*Then this property is shared by all products of the $A_i$.*

*Proof.* We prove the lemma by induction. It is enough to consider the products of two generic matrices $A = (A^i_j)$, $B = (B^i_j)$, satisfying the hypotheses.

Assume first that $k = 0$ for $B$, namely $\sum_{k \neq \ell} B^i_k \geq B^i_\ell$ for all $i, \ell$. Then

$$\sum_{k \neq \ell} (AB)^i_k = \sum_{\substack{1 \leq j \leq n \\ k \neq \ell}} A^i_j B^j_k = \sum_{1 \leq j \leq n} A^i_j \sum_{k \neq \ell} B^j_k \geq \sum_{1 \leq j \leq n} A^i_j B^j_\ell = (AB)^i_\ell.$$

Assume now that $k > 1$ for $B$ and denote by $I = (i_1, \dots, i_k)$ the rows with a 1 and all other entries equal to 0. Every line (if any) of $A$ with a 1 and all other entries equal to 0 leaves unaltered the corresponding row in $B$ and therefore the new line satisfies the conditions in the theorem. Otherwise we notice that

$$\sum_{k \neq \ell} (AB)^i_k = \sum_{\substack{1 \leq j \leq n \\ k \neq \ell}} A^i_j B^j_k \geq \sum_{\substack{1 \leq j \leq n \\ j \notin I}} A^i_j B^j_\ell + \sum_{\substack{1 \leq j \leq n \\ j \in I}} A^i_j \sum_{k \neq \ell} B^j_k.$$

If $j \in I$ then $\sum_{k \neq \ell} B^j_k$ is either 0 or 1. Since by hypothesis there are at least two such rows and $\sum_{k \neq \ell} B^j_k + \sum_{k \neq \ell} B^{j'}_k \geq 1$ for every $j, j' \in I$, $j \neq j'$, and

the corresponding entries $A^i_j$ and $A^i_{j'}$ are both equal to 1, then

$$\sum_{\substack{1\leq j\leq n\\ j\in I}} A^i_j \sum_{k\neq\ell} B^j_k \geq \sum_{\substack{1\leq j\leq n\\ j\in I}} A^i_j B^j_\ell = (AB)^i_\ell,$$

therefore $\sum_{k\neq\ell}(AB)^i_k \geq (AB)^i_\ell$. □

**Proposition 16.** *The Apollonian semigroup is a fast gasket with coefficient* $c \geq 1/4$.

*Proof.* Note first of all that Hirst matrices satisfy previous Lemma's conditions. Moreover the entries in the third line are not smaller than all other entries in the same column and it is easy to see by induction that this property is preserved by products.

Let $\|A\|_\infty = \max_{1\leq i\leq n}\sum_{1\leq j\leq n}|A^i_j|$. A look at the 6 matrices $H_{ij}$, $i\neq j$, shows that their third column has always at least three non-zero entries, so that $\|A_{IJ}\|_\infty \geq \|A_J\|_\infty \sum_{j\neq j_0}|A^i_3|$ where $j_0$ is the index of the element of the third column (if any) equal to zero (otherwise just set $j_0 = 1$). By the previous Lemma and the fact that the norm of every $A\in\boldsymbol{H}$ in concentrated in the third row, the sum of any three entries of the third row of $A$ is always larger than $\|A\|$, so that $\|A_{IJ}\|_\infty \geq \|A_J\|_\infty\|A_I\|$. Since $4\|A\| \geq \|A\|_\infty \geq \|A\|$, the claim follows. □

Analytical bounds for the exponent $s_{\boldsymbol{H}}$ of the Apollonian semigroup were studied in detail by Boyd in [Boy70, Boy72, Boy73a] and we do not attempt to improve them here.

Increasingly accurate *numerical* evaluations of $s_{\boldsymbol{H}}$ with several different techniques have been given over the last half-century by Melzak [Mel69], Boyd [Boy82], Manna and Herrmann [MH91], Thomas and Dhar [TD94] and McMullen [McM98] giving respectively the following values, with a *heuristic* error of 1 unit on the last digit: 1.306951, 1.3056, 1.30568, 1.30568673, 1.305688. We remark that, among all these evaluations, the one with the largest number of digits, given by Thomas and Dhar, is the only one based on a *heuristic* method, while the others are based on *exact* methods.

Partly to test our own software evaluating the function $N_{\boldsymbol{H}}(k)$ for a generic gasket $\boldsymbol{H}$ and partly because the computational power of computers increased quite a lot over the last fifteen years, which is how old is the last evaluation of the exponent, we repeated the elementary evaluation made

| | |
|---|---|
| $\boldsymbol{C}_2$ (19) | 3, 15, 71, 287, 1231, 4911, 19831, 79279, 318383, 1273807, 5098247, 20391887, 81590055, 326364583, 1305483999, 5221928631, 20888160751, 83552534287, 334211194663 |
| $\boldsymbol{C}_3$ (13) | 4, 22, 148, 760, 4594, 24646, 136372, 740650, 4046188, 22022770, 119929126, 652445212, 3550689778 |
| $\boldsymbol{C}_4$ (12) | 5, 37, 293, 2197, 15125, 103669, 714245, 4849045, 32901077, 222724789,1507986917, 10202765749 |
| $\boldsymbol{A}_3$ (13) | 3, 12, 64, 316, 1784, 10004, 58224, 341386, 2033906, 12170708, 73208110, 441772966, 267292497 |
| $\boldsymbol{H}$ (39) | 0, 1, 3, 8, 18, 48, 113, 278, 681, 1722, 4238, 10488, 25927, 64086, 158266, 391062, 967315, 2390800, 5909752, 14608522, 36115118, 89275994, 220684802, 545546400, 1348603780, 3333755028, 8241076212, 20372155276, 50360227721, 124491161884, 307744098990, 760747405278, 1880578271904, 4648814463680, 11491932849933, 28408221038996, 70225503797745, 173598409768852, 429137646728801 |
| $\boldsymbol{F}$ (35) | 2, 7, 16, 34, 84, 151, 348, 679, 1546, 3034, 6546, 13476, 28409, 59578, 122139, 261698, 531191, 1144823, 2314772, 4986951, 10132768, 21667197, 44400099, 94074745, 194587388, 408651488, 852101402, 1777247239, 3726410796, 7738675037, 16274400897, 33739772516, 71002774691, 147235829060, 309533001058 |

Table 3: Values of $N_{\boldsymbol{A}}(2^k)$ for small $k$ for the *cubic semigroups* $\boldsymbol{C}_i$, $i = 2, 3, 4$, the Apollonian semigroup $\boldsymbol{A}_3$, the Hirst semigroup $\boldsymbol{H}$ and the semigroup $\boldsymbol{F}$ of Example 10. In the left column it is also reported the numnber of terms displayed in the right one.

by Boyd in 1982 by evaluating $N_{\boldsymbol{H}}(k)$ for $k = 2^p$, $p = 1, \dots, 40$ with respect to the norm $\|A_I\| = \sum_{1 \le i,j \le 4} (A_I)_{ij} v^i w^i$, where $v = (-1, 2, 2, 0)$ and $w = (1, 1, 1, 2)$ (this way $\|A_I\|$ is equal to the the curvature of the circle of multi-index $I$ in the Apollonian gasket generated by the circles of radius $-1, 2, 2$), and then interpolating the data obtained (see Table 3). We found a value of $s_{\boldsymbol{H}} \simeq 1.30568673$ which fully confirms the heuristic evaluation of Thomas and Dhar and suggests an error of 2 on the last digit of the estimate of McMullen.

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